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Tensor invariants and their applications to quantum information theory.

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NORMATIVE REFERENCES

This dissertation uses references to the following standards:

1 Law of the Republic of Kazakhstan “About science” dated 02/18/2011 N2 407-IV LRK.

2 Acting Order Minister of Health and Social Development of the Republic of Kazakhstan dated July 31, 2015 No. 647 approval of state mandatory standards and standard professional educational programs in medical and pharmaceutical specialties” as amended by the Order of the Minister of Health of the Republic of Kazakhstan dated February 21, 2020 N9 RK DSM-12/ 2020. Registered with the Ministry of Justice of the Republic of Kazakhstan on February 27, 2020 No. 20071.

3 Rules for awarding academic degrees approved by order of the Minister of Education and Science of the Republic of Kazakhstan dated March 31, 2011 127 (registered in the Register of State Registration of Normative Legal Acts under No. 6951).

4 GOST 7.32-2017 Interstate Standard. System of standards on information, librarianship and publishing. Research report. Structure and design rules.

DEFINITIONS

In this dissertation the following terms with corresponding definitions are used:

Hypermatrices	A hypermatrix is a generalization of a matrix to higher dimensions, represented as a multidimensional array of elements indexed by multiple indices.
Tensor	A tensor is a multidimensional generalization of a vector or linear operator. In the fixed basis, tensor is represented with hypermatrix.
Invariants	are properties of a mathematical object that remain unchanged under a set of transformations or operations.
Group	is a set equipped with a single binary operation that satisfies four conditions: closure, associativity, the existence of an identity element, and the existence of inverse elements for every element in the set.
Group action	is a rule by which group interact with given set group acts on.
Group orbit	is the set of all elements that can be reached by applying every element of a group to a particular element of a set.
Quantum entanglement	is a physical phenomenon in quantum mechanics where pairs or groups of particles become interconnected, such that the state of one particle cannot be described independently of the state of the others, even when separated by large distances.
Quantum information theory	is a field of study that investigates how quantum systems can be used to represent, process, and transmit information. It extends classical information theory by leveraging the principles of quantum mechanics, such as superposition, entanglement, and quantum measurement, to develop new computational and communication protocols that can outperform their classical counterparts. This field underpins technologies such as quantum computing, quantum cryptography, and quantum communication.

NOTATIONS AND ABBREVIATIONS

In this dissertation the following notations and abbreviations are used:

$\mathbb{N}$	Set of natural numbers.
$\mathbb{Z}$	Set of integer numbers.
$\mathbb{R}$	Set of real numbers.
$\mathbb{C}$	Set of complex numbers.
$\otimes$	Tensor product (or Kronecker product).
dim	Dimension of a vector space.
$\lambda \vdash n$	partition λ of number n.
$\cong$	Is isomorphic.
$\mathbb{C}[V]$	ring of polynomial functions over vector space V with complex coefficients.
GCT	Geometric Complexity Theory.
ACT	Algebraic Complexity Theory.
GL(n)	General Linear group of order n over $\mathbb{C}$.
SL(n)	Special Linear group of order n over $\mathbb{C}$.
$sl_n(\mathbb{C})$	Special Linear Lie group of order n over $\mathbb{C}$.
SLOCC	Stochastic local operations and classical communication.

INTRODUCTION

One of the rapidly growing sections of Computer Science, is the one intersecting with Quantum Entanglement phenomena. For instance, potentially, quantum computers may offer more efficient algorithms, quantum cryptography may offer more secure protocols and so on. For this to happen, there is an urge of understanding the structure of quantum system states on a deep level, which may be called one of the biggest problems of this century.

One can represent quantum system state as tensor. **Tensors**, also known as multidimensional arrays or hypermatrices in computer science, are indispensable components in various areas, such as mathematics, physics, computer science and data analysis, offering a versatile framework for organizing and analyzing complex data structures. Their importance spans across various computational domains, from quantum computing to computational complexity and machine learning and beyond.

The role of the linear algebra is crucial in algorithms of machine learning and data analysis. For instance, well known linear regression method is formulated in terms matrices and vectors, and solution relies on methods of linear algebra. Unfortunately, this is not easily generalized for higher dimensions, if one replace matrices with tensors.

This demonstrates the urge to understand fundamental properties of tensors better and to provide the ground for methods consistent with theory. The theory of multilinear algebra, where tensor is central object, is the hot topic in scientific world in the last few decades, especially in quantum computing, complexity theory, statistical learning theory, signal processing, and data analysis to problems in geometry and representation theory.

One of aspects of tensor theory that is not well studied is the symmetries and orbits of tensors under different group actions. Such questions are particularly important in quantum information theory and geometric complexity theory. Formally, let $V = (\mathbb{C}^n)^{\otimes d}$ be the tensor space with the multilinear action of a d-tuple of special linear groups $\mathrm{SL}(n)^{\times d}$. For the given tensors $T_1, T_2 \in V$ is there an element $G \in \mathrm{SL}(n)^{\times d}$ s.t. $T_1 = G \cdot T_2$?

For $d = 2$ this question reduces to determining the rank of a matrices T_1 and T_2. While for $d > 2$, the question gets complicated and there is no satisfying criteria of this to happen. The problem of determining the negative answer to this problem is called "Orbit separation problem". To tackle this problem one can study polynomials that are constant on the orbits of the group action, called invariant polynomial. Then if such polynomial $P(X)$ is provided, then $P(T_1) = 0$ and $P(T_2) \neq 0$ implies that the tensors are in different orbits. The problem, is that as n or d grows, the size of such polynomial grows exponentially. So it is essential to obtain the smallest invariants. The goal of this work is to provide systematic study of such polynomials and provide a clear way of generating invariants of the smallest degrees in various cases.

Relevance of the work. The work on combinatorial properties and invariants of hypermatrices is of great significance not only in mathematics in general, but also in the ever-evolving field of quantum computing. As quantum information theory continues to advance, the need to understand the fundamental structures and properties of hypermatrices becomes increasingly important.

In quantum information theory, a fundamental concept to grasp is entanglement. Quantum entanglement serves as a crucial resource in quantum information processing. The main question revolves around how to measure and classify the entanglement of quantum states. Polynomial functions derived from the coefficients of pure states, which remain unchanged (invariant) under *stochastic local operations and classical communication* (SLOCC) transformations, have been extensively studied and utilized in constructing measures of entanglement.

Another fundamental importance of this research arise in the algebraic and geometric complexity theory (ACT and GCT). Geometric complexity theory is an approach via algebraic geometry and representation theory towards the P vs. NP and related problems.

Therefore, the exploration of combinatorial properties and invariants of hypermatrices is not only academically stimulating but also holds profound practical implications for the future of quantum computing.

Object of research. Let $V = \mathbb{C}^n \otimes \ldots \otimes \mathbb{C}^n$ (d times) be the tensor space. Under the fixed basis, each element $T \in V$ can be represented with the hypermatrix as

$$T = \sum_{1 \leqslant i_1, \ldots, i_d \leqslant n} T_{i_1, \ldots, i_d} e_{i_1} \otimes \ldots \otimes e_{i_d}. \tag{1}$$

Let the *special linear group* be the group of invertible transformations $\mathrm{SL}(n) := \mathrm{SL}(n, \mathbb{C}) = \{A \mid A \text{ is complex } n \times n \text{ matrix}, \det(A) = 1\}$. The group $\mathrm{SL}(n)^{\times d}$ acts on simple tensors as follows:

$$(A_1, \ldots, A_d) \cdot v_1 \otimes \ldots \otimes v_d = A_1 v_1 \otimes \ldots \otimes A_d v_d \tag{2}$$

and extended linearly, where $A_i \in \mathrm{SL}(n)$ and $v_i \in \mathbb{C}^n$. By $\mathbb{C}[V]$ the ring of polynomials with entries of hypermatrix as variables is denoted. By $\mathbb{C}[V]_m$ degree-m part of polynomial ring and by $\mathbb{C}[V]^G$ the subspace (subring) of polynomials invariant under the group G-action is denoted. Let *invariant polynomials* be the elements of $\mathbb{C}[V]_m^{\mathrm{SL}(n)^{\times d}}$.

The Kronecker coefficients $g(\lambda^{(1)}, \ldots, \lambda^{(d)})$ are the structural constants of tensor product of irreducible representations, where $\lambda^{(i)}$s are partitions. It is also an important object of the research, since for $\lambda^{(i)} = \lambda = k^n$

$$g(k^n, \ldots, k^n) = \dim \mathrm{C}[(\mathbb{C}^n)^{\otimes d}]_{nk}^{\mathrm{SL}(n)^{\times d}}, \tag{3}$$

i.e. the problem of determining the existence reduces to the problem of determining positivity of Kronecker coefficients.

Subject of research. The subject of research are polynomial invariants of minimal possible degree for odd d, thus, in particular, positivity of Kronecker coefficients. Also, the computational aspects of invariant polynomials.

Goal of the research. The goal of the work is to develop the framework to study invariant polynomials of tensors, especially ones of minimal degree, and study related objects like highest weight vectors and Kronecker coefficients. The aim is to determine minimal possible degree of invariant polynomials for various n and odd d. Also, the aim is to develop an algorithm for generating invariant polynomials.

Problems of the dissertation.

1 Systematic research of fundamental properties of invariant polynomials and related objects;
2 determine minimal degree for which invariant exist;
3 describe all invariant polynomials;
4 calculate all coefficients of each invariant polynomial;
5 develop the algorithm to compute the basis of invariant polynomials;
6 compute invariant polynomials in all feasible cases.

Scientific novelty. The results obtained and published have the following scientific contribution:

1 a novel methodology to study invariant polynomials is developed, which provides a new systematic way to study invariant polynomials;
2 the formula of the generic invariant polynomial of fixed degree is obtained;
3 new lower bound on the minimal degree of invariant polynomial is obtained, which is sharp and archived in fundamental cases;
4 novel algorithm of calculation of the basis of invariant polynomials is presented, which exponentially faster than all naive approaches;
5 concrete calculation of the basis of invariant polynomials of fixed degree in all feasible cases is presented, which is the new result at least in cases of 7 qubits and 5 qutrits.

Research methods. The solution to the given problem was obtained by algebraic and combinatorial methods. The proofs extensively use Schur-Weyl duality and representation theory. Practical results were held using programming language C++. Also, Sage library on Python were used to compute tables of Kronecker coefficients.

Practical significance. Concrete invariant polynomials of minimal degree are obtained at least in the following cases: $(d,n) = (3,2)$, $(d,n) = (3,3)$, $(d,n) = (5,2)$, $(d,n) = (7,2)$, $(d,n) = (5,3)$.

Practical significance of work is interdisciplinary. In quantum information theory and quantum computing the research helps to understand better the levels of entanglement of quantum systems.

In theoretical computer science the study of invariant polynomial theory provides a framework to develop faster algorithms, for instance algorithm of matrix multiplication.

Developed methods and results provide polynomials of minimal degree which are feasible to compute, which is the huge challenge in specified areas.

Submitted for the defense:

1 Systematic research of invariant polynomials of tensors, which offers unified combinatorial framework to study invariants of any degree. This includes: derivation of concrete formula of generic invariant polynomial, derivation of the spanning set of invarants of fixed degree, vanishing conditions, general properties and important coefficients.

2 Calculation of the lower bound for the minimal degree of invariant polynomials of given tensor space, which is shown to be sharp in fundamental cases.

3 Design of efficient algorithm for calculation of the basis of invariant polynomials of fixed degree, which exponentially faster that all naive approaches.

4 Calculation of the basis of invariant polynomials of fixed degree in all feasible cases: degree 4 and degree 6 for 3 qubits, degree 6 and degree 9 for 3 qutrits, degree 4 and degree 6 for 5 qubits, degree 6 and degree 9 for 5 qutrits, degree 4 of 7 qubits.

Approbation of obtained results. PhD thesis results are published the following international journals:

1 **Forum of Mathematics Sigma.** - Vol. 11. - 2023. - P. e63. (Scopus Q1, Web of Science Q1).

2 **International Mathematics Research Notices.** - 2023. - Vol. 2023, no. 20. - 17552-17599. (Scopus Q1, Web of Science Q2).

3 **Linear Algebra and its Applications.** - 2023. - Vol. 656. - Pp. 224–246 (Scopus Q1, Web of Science Q1).

4 **Proceedings of the American Mathematical Society.** - 2022. - Vol. 150, no. 10. - Pp. 4113–4128 (Scopus Q1, Web of Science Q2).

5 **Herald of the KBTU.** - 2024. - Vol. 21(2). - Pp. 95–105 (KKSON).

6 **Herald of the KBTU.** - 2024. - Vol. 21(3). - Pp. 58–65 (KKSON).

7 **Herald of the KBTU.** - 2024. - Vol. 21(3). - Pp. 128–136 (KKSON).

Also, the talks were reported at:

1 *Invariant polynomials with applications to Quantum computing*, 2024 IEEE

AITU: Digital Generation, **Astana IT University**, 2024.

2 *Fundamental invariants*, the Thematic Research Programme "Tensors: geometry, complexity and quantum entanglement", University of Warsaw, Excellence Initiative–Research University, the **Simons Foundation**, **Polish Academy of Sciences**, 2023.

3 *Some unimodal sequences of Kronecker coefficients*, Seminar in Partition Theory, q-Series and Related Topics, **Michigan Technological University**, 2024.

4 *Fundamental invariants*, **Polish Academy of Sciences seminar**, Bendlewo, Poland, 2023.

5 *MacMahon's statistics on higher-dimensional partitions*, **ADA University seminar**, 2022.

6 *Combinatorial hyperdeterminant and multiplanar networks*, Traditional international April mathematical conference dedicated to the Day of Science Workers of the Republic of Kazakhstan, **Institute of mathematics, Kazakhstan**, 2022.

1 LITERATURE REVIEW

Tensors, fundamental in mathematics and physics, serve as a sophisticated framework capable of representing laws and properties in a form invariant under coordinate transformations. They are integral in the formulation of tensor calculus, which supports the general theory of relativity and the representation of physical laws in continuum mechanics.

1.1 Tensors in Computer Science

Tensors play a pivotal role in computer science, especially in areas such as data analysis, algorithm design, and computational complexity. Their multi-dimensional structure enables efficient manipulation and storage of large datasets, which are common in machine learning, computer vision, and other computational fields.

In computer science, tensors are employed as advanced data structures that generalize matrices to higher dimensions. This generalization is crucial for handling data with multiple attributes efficiently. Tensors facilitate operations such as tensor decomposition and tensor-based data completion, which are fundamental in algorithms for image processing, video analysis, and neural networks [1].

Tensors also enhance the computational capabilities in algorithms. They are integral in designing algorithms for tensor decompositions and multilinear operations, which are used in quantum computing, symbolic computation, and optimization tasks. The complexity of tensor operations often dictates the efficiency of algorithms in practical applications, making the study of tensor-based algorithms essential for advancing computational theories and practices [2].

Invariant polynomials in computer science are used to simplify tensor expressions and to identify properties of tensors that are invariant under certain transformations, such as rotations or permutations. These invariants are crucial in computer algebra systems, where they help in simplifying expressions involving tensors and in solving systems of polynomial equations. For instance, invariant theory has been applied to Riemann tensors in general relativity to simplify and classify tensor expressions algorithmically [3].

In computational geometry and computer vision, tensors and their invariant properties are used to identify and classify geometric shapes and to solve computer vision problems such as object recognition and image segmentation. The use of algebraic invariants derived from tensor expressions facilitates the handling of geometric transformations and comparisons of multidimensional shapes [4].

1.2 Tensors in mathematics

Tensors, or hypermatrices, are fundamental mathematical structures extending beyond matrices to higher dimensions. They are essential in various branches of mathematics such as algebra, differential geometry, and functional analysis.

Tensors serve as the backbone for many mathematical theories. They are crucial in tensor calculus, which generalizes vector calculus to higher dimensions, allowing for the manipulation of multidimensional data in fields like differential geometry and complex manifolds. Tensors are also pivotal in representing and solving systems of linear equations where variables interact across multiple dimensions, making them indispensable in mathematical physics and engineering [5].

In algebra, tensors are studied for their properties under various transformations, particularly in the context of symmetry groups and invariant theory. Polynomial invariants of tensors are crucial in classifying tensor types under group actions and understanding their algebraic and geometric properties. These invariants are used to solve polynomial equations that remain unchanged under group actions, providing insights into the symmetry and conservation laws in physics and other sciences [6].

Invariant polynomials are significant in understanding the structural properties of tensors. They help define tensor ranks and dimensions, classify tensor types, and explore tensor decompositions. For example, the E-characteristic polynomial, a type of invariant polynomial, helps in determining the eigenvalues of tensors, which are critical in many applications including the stability analysis of systems [7]. These invariant polynomials are also essential in the study of tensor symmetries and tensor field theories, providing a deep understanding of the physical and abstract spaces tensors operate within.

From a computational perspective, tensors and their polynomial invariants are used in algorithms for tensor decompositions and numerical simulations in high-dimensional spaces. These decompositions simplify complex tensor operations, making them manageable and applicable in practical scenarios like signal processing and quantum computing. The study of tensors from a computational angle often involves exploring efficient ways to handle large-scale tensor data, optimize tensor networks, and improve algorithmic performance in tensor calculations [8].

1.3 Tensors in Machine Learning

Tensors play a significant role in machine learning by providing a way to generalize matrix operations to higher-dimensional spaces. Their application spans various aspects of machine learning, from basic operations in neural networks to complex structures in deep learning architectures.

In machine learning, tensors are utilized primarily for their ability to represent and process multi-dimensional data efficiently. For example, in neural networks, tensors are used to store and manipulate data for weights, biases, and activations, facilitating operations that are inherently multi-dimensional, such as convolutions in convolutional neural networks. Tensor operations help in optimizing computations and leveraging hardware accelerations, such as GPUs and TPUs, for faster processing [9].

Tensor decomposition techniques, such as CANDECOMP/PARAFAC (CP) and Tucker decomposition, are critical in simplifying complex tensor operations and extracting meaningful features from multi-dimensional data. These techniques are used in unsupervised learning scenarios, like clustering and anomaly detection, where identifying inherent structures in data is crucial. Tensor decomposition helps in breaking down high-dimensional tensors into simpler, interpretable components, making it easier to analyze and visualize data [10].

Deep learning frameworks have integrated tensor operations deeply into their core, allowing for the design of complex models such as deep tensor neural networks. These models use tensor operations to manage and learn from data that have natural multi-dimensional structures, such as images, videos, and various forms of sequential data. The integration of tensor operations in deep learning not only improves the model's performance but also enhances its ability to generalize across different tasks [11].

Recently, tensor methods have found applications in quantum machine learning, where they are used to handle complex quantum data structures. In this setting, tensors help represent quantum states and operations efficiently, bridging the gap between classical machine learning techniques and quantum computing. This cross-disciplinary approach is poised to leverage the computational advantages of quantum computing, potentially leading to breakthroughs in machine learning capabilities [12].

1.4 Tensors in Quantum Information Theory

Tensors, particularly tensor networks, have become instrumental in the fields of quantum physics and quantum information theory. These tools are pivotal for modeling and understanding complex quantum systems, where traditional computational methods are often inadequate.

In quantum physics, tensor networks are employed to represent quantum states, particularly in many-body quantum physics. These networks, such as matrix product states (MPS), projected entangled pair states (PEPS), and the multiscale entanglement renormalization ansatz (MERA), provide a structured way to express the wavefunctions of quantum systems. They allow the efficient approximation of quantum states with complex entanglement structures, making them valuable for studying systems that are otherwise computationally prohibitive due to the exponential growth of the Hilbert space with the number of particles [13].

Tensor networks facilitate the simulation of quantum dynamics, where they help in solving the Schrödinger equation for systems with large numbers of particles. This is particularly relevant in condensed matter physics and quantum chemistry, where understanding the properties of materials at the quantum level is crucial. Tensor networks reduce the complexity of quantum simulations by decomposing high-dimensional tensors into networks of lower-dimensional systems, thus enabling practical simulations of phenomena like quantum phase transitions and quantum entanglement [14].

In quantum information theory, tensor networks are used to analyze and optimize quantum circuits, entanglement, and communication protocols. They provide a visual and mathematical language to describe quantum processes, including teleportation, quantum error correction, and the implementation of quantum algorithms. Tensor networks help in designing more efficient quantum circuits by minimizing resource requirements such as the number of qubits and quantum gates, thus aiding in the development of scalable quantum computing technologies [12].

The interface between tensor networks and machine learning, particularly quantum machine learning, is an emerging area of research. Tensor networks are being used to develop new quantum machine learning models that can potentially run on quantum computers. These models aim to leverage the quantum properties of superposition and entanglement to solve machine learning problems more efficiently than classical computers. The synergy between tensor network theory and quantum algorithms offers promising avenues for breakthroughs in both fields [15].

1.5 Polynomial invariants

Arthur Cayley's pioneering works [16, 17] on hyperdeterminants has been a cornerstone in the development of higher-dimensional algebraic structures.

Further, the work of Gelfand, Kapranov, and Zelevinski [18] on hyperdeterminants has had a profound impact on multiple areas of mathematics, including algebraic geometry, combinatorics, and mathematical physics. Their research is most notably recognized for its rigorous formalization and expansion of the concept of hyperdeterminants and its implications for systems of equations and geometric structures. Ottaviani (2013) offers a comprehensive introduction to hyperdeterminants, emphasizing their historical development and mathematical significance. This survey is instrumental in making the complex concepts accessible, bridging classical determinant theory with the more generalized GKZ hyperdeterminants [19].

Yao, Liu, and Bu (2015) extend the tensor product defined by GKZ, demon-

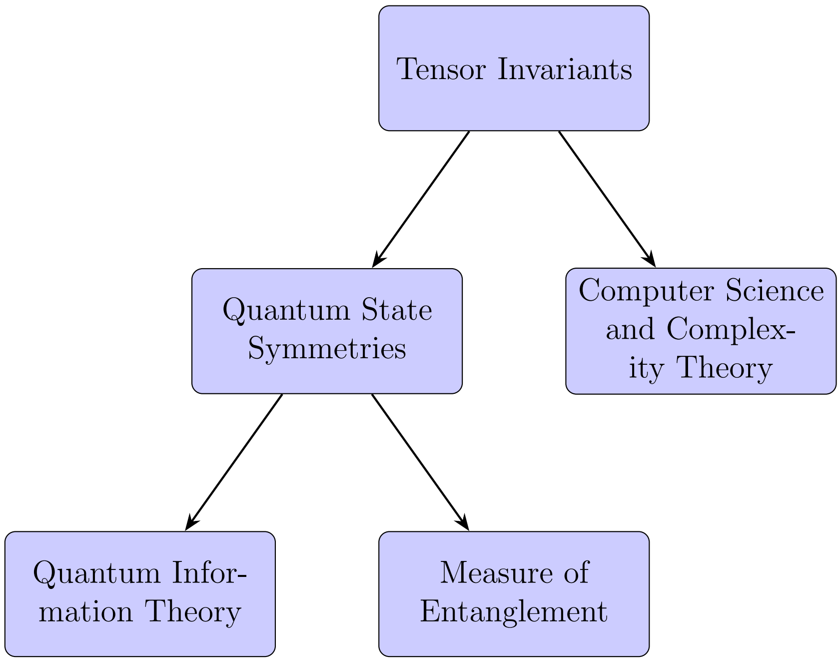

Figure 1.1 – Diagram representing applications of Tensor invariants

strating its associative properties and applying it to the calculation of hyperdeterminants for specific tensor formats. This work showcases the utility of GKZ's foundational concepts in enhancing our understanding of tensor algebra and its applications in complex systems [20].

Weyman (1994) uses homological algebra to expand on GKZ's framework, particularly in the computation of discriminants and hyperdeterminants via higher direct images. This paper effectively applies GKZ's theoretical framework to solve classical problems in algebraic geometry, providing deeper insights into the structures of singularities and dual varieties [21].

Brylinski and Brylinski (2000) explore polynomial functions on tensor states in $(\mathbb{C}^n)^{\otimes k}$ that are invariant under $SU(n)^k$. They describe the space of these invariant polynomials in terms of symmetric group representations, providing a generalization of the determinant for even k and fixed n and d, with detailed study on the behavior of these polynomials in higher dimensions [22].

Allman et al. (2012) discuss tensors of rank n over $\mathbb{C}$, constructing polynomial invariants under group actions that distinguish certain tensor orbits from points in their closures. This study extends the hyperdeterminant concept beyond the classical $n = 2$ case, showing how these invariants help in understanding the algebraic structure of tensors [23].

Qi (2005) defines symmetric hyperdeterminants and eigenvalues for real supersymmetric tensors. This paper not only explores the roots of these characteristic polynomials but also relates them to the tensor's hyperdeterminant, providing insights into the tensor's algebraic and geometric properties [24].

Manon and Zhou (2012) delve into semigroup algebras tied to the decomposition of tensor products under $sl_3(\mathbb{C})$, providing new insights into the combinatorial structure of these polynomials through the lens of algebraic geometry and representation theory [25].

This study by Endrejat and Buettner (2006) compares the polynomial invariants for four qubits, introduced by Luque and Thibon, with optimized Bell inequalities and a combination of two qubit concurrences. Their findings highlight the utility of these polynomial invariants in measuring genuine 4-qubit entanglement across various parameter-dependent states from different SLOCC classes, demonstrating their practical applications in quantum information processing [26].

Gour and Wallach (2012) delve into the generating set of SL-invariant polynomials in four qubits that are also invariant under permutations of the qubits. They identify four critical polynomials of varying degrees which play a pivotal role in characterizing the algebraic variety of these states, showing the mathematical beauty and complexity of four-qubit systems [27].

Sharma and Sharma (2012) use local unitary invariance and negativity fonts to construct and identify polynomial invariants of various degrees. They propose a classification of four-qubit states into seven major classes based on the nature of their correlations, providing a structured approach to understanding four-qubit entanglement [28].

Entanglement in Four Qubit States: Jafarizadeh et al. (2019) explore polynomial entanglement measures of degree 2 for even-N qubits X states. They plot a hierarchy of entanglement classification for four qubit pure states using new invariants, highlighting the connection between entanglement measures, GM concurrence, and one-tangle [29].

Sharma and Sharma (2014) discuss an inductive process for constructing local unitary invariant polynomials for multipartite quantum states. This method aids in understanding and quantifying non-local N-way correlations in N-qubit pure states, highlighting the broader applicability of polynomial invariants across various qubit configurations [30].

Teodorescu-Frumosu and Jaeger (2003) explore quantum Lorentz-group invariants for n-qubit systems, using a real (Lorentz) group to describe the action of SLOCC operations. This study provides a natural quantum Lorentz-group invariant length that can describe entanglement in terms of Minkowskian analogs to quantum state purity [31].

Gao, Yan, and van Enk (2014) discuss the "permutationally invariant part of a density matrix" and its applications for quantifying and characterizing entanglement in N-qubit systems. This concept allows for the establishment of a hierarchy of multipartite separability criteria and provides a measure of entanglement that is efficiently measurable [32].

Heshmati et al. (2019) illustrate that the polynomial invariant of degree 2 is experimentally measurable for even N-partite pure states of qubits. This work underscores the physical interpretation and experimental accessibility of polynomial invariants in quantum information theory [33].

Li and Li (2013) provide a method for constructing SLOCC polynomial invariants of degree 6 for even n-qubit states. Their approach offers a practical tool for classifying entangled states under SLOCC, highlighting the role of polynomial invariants in understanding the complex structure of entanglement [34].

2 THEORETICAL FOUNDATIONS OF TENSOR INVARIANTS

2.1 Preliminaries

2.1.1 Tensors and Hypermatrices

a) Tensors

Tensor product of two vector spaces A and B is the quotient space $A \otimes B := A \times B/R$, where $A \times B$ is the Cartesian product of vector spaces and R is the vector subspace of $A \times B$ spanned by the elements of the form: $(a_1 + a_2, b) - (a_1, b) - (a_2, b)$, $(a, b_1 + b_2) - (a, b_1) - (a, b_2)$, $(sa, b) - s(a, b)$, $(a, sb) - s(a, b)$, for any $s \in \mathbb{C}$, $a_1, a_2, a \in A$ and $b_1, b_2, b \in B$. Then the tensors are the elements of tensor product space. Attention is focused on specific tensor space $(\mathbb{C}^n)^{\otimes d} := \mathbb{C}^n \otimes \ldots \otimes \mathbb{C}^n$ (repeated d times).

b) Hypermatrices

A d-dimensional hypermatrix is a d-dimenisonal array $T = (T_{i_1,\ldots,i_d})$ with $i_j \geqslant 1$. Each hypermatrix display coordinates of the element

$$\sum_{i_1,\ldots,i_d \geqslant 1} T_{i_1,\ldots,i_d} e_{i_1,\ldots,i_d} = T \in (\mathbb{C}^n)^{\otimes d}$$

and each element of that space can be represented as hypermatrix. For $i = (i_1, \ldots, i_d) \in [n]^d$ here is written $e_i = e_{i_1,\ldots,i_d} = e_{i_1} \otimes \ldots \otimes e_{i_d}$.

A hypermatrix T is called $\mathbb{N}$-*hypermatrix* if it has finite support $\operatorname{supp}(T) = \{\mathbf{i} = (i_1, \ldots, i_d) : T_{i_1,\ldots,i_d} \neq 0\}$ and $T_{i_1,\ldots,i_d} \in \mathbb{Z}_{\geqslant 0}$. If additionally $T_{i_1,\ldots,i_d} \in \{0, 1\}$, then T is called $(0,1)$-*hypermatrix*. In paper, hypermatrix is referred to as integer valued array.

A d-tuple of vectors $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(d)})$ is called *marginals* of T whenever

$$\alpha_j^{(\ell)} = \sum_{i_1,\ldots,\widehat{i_\ell},\ldots,i_d} T_{i_1,\ldots,j,\ldots,i_d}.$$

for each $i \in [d], \ell \geqslant 1$. In other words, $\alpha_j^{(\ell)}$ is equal to sum of elements in j-th slice of ℓ-th direction.

Let $m = |\operatorname{supp}(T)|$. For d-dimensional hypermatrix T, a *d-line notation* of T is the $d \times m$ table having columnn $(i_1, \ldots, i_d)$ exactly $T_{i_1,\ldots,i_d}$ times and columns are ordered lexicographically.

2.1.2 Partitions

A *partition* is a sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of positive integers $\lambda_1 \geqslant \cdots \geqslant \lambda_\ell$, where $\ell(\lambda) = \ell$ is its *length*. The *size* of λ is $\lambda_1 + \ldots + \lambda_\ell$. Every partition λ can be represented as the *Young diagram* $\{(i, j) : i \in [1, \ell], j \in [1, \lambda_i]\}$. By λ' is denoted the *conjugate* partition of λ whose diagram is transposed. Rectangular partitions are denoted as $a \times b := (\underbrace{b, \ldots, b}_{a \text{ times}})$. For $\lambda \subseteq a \times b$ we also denote by $a \times b -_a \lambda := (b - \lambda_a, \ldots, b - \lambda_1)$ the *complementary* partition of λ inside $a \times b$.

2.1.3 Highest weight vectors

All the groups considered are over $\mathbb{C}$. Let the group $\mathrm{GL}(n)$ acts diagonally on $\otimes^m \mathbb{C}^n$ by left multiplication. Fix the standard basis of $\mathbb{C}^n = \langle e_1, \ldots, e_m \rangle$, then the standard torus $T(n) := (\mathbb{C}^\times)^n \subseteq \mathrm{GL}(n)$, the set of non-degenerate diagonal matrices, give rise to the weight decomposition

$$\bigotimes^m \mathbb{C}^n = \bigoplus_{\alpha \in \mathbb{Z}^n} W_\alpha$$

where $\alpha \in \mathbb{Z}^n$ and $W_\alpha = \{v \in \bigotimes^n \mathbb{C}^n \mid t \cdot v = t^\alpha v = t_1^{\alpha_1} \ldots t_\ell^{\alpha_\ell} v\}$. The *highest weight vector* of weight λ is nonzero weight vector of weight λ which is invariant under the action of subgroup of unitriangular matrices $U(n) \subset \mathrm{GL}(n)^{\times d}$, i.e. $U(n) \cdot v = v$ and $v \in W_\lambda$.

For n^d-dimensional vector space $(\mathbb{C}^n)^{\otimes d} = \langle e_{i_1,\ldots,i_d} \rangle_{1 \leqslant i_j \leqslant n}$ (where $e_{i_1,\ldots,i_d} = e_{i_1} \otimes \ldots \otimes e_{i_d}$) there is the natural (multilinear) action of the group $\mathrm{GL}(n)^{\times d}$ given by

$$(G_1, \ldots, G_d) \cdot v_1 \otimes \ldots \otimes v_d = G_1 v_1 \otimes \ldots \otimes G_d v_d$$

where $v_i \in \mathbb{C}^n$ and $G_i \in \mathrm{GL}(n)$. This action induces the action on the tensor space $\bigotimes^m (\mathbb{C}^n)^{\otimes d}$. For group $\mathrm{GL}(n)^{\times d}$ the definitions above can be rewritten similarly, where $T(n)$ is replaced with $T(n)^{\times d}$, $U(n)$ with $U(n)^{\times d}$ and the weights $\alpha \in \mathbb{Z}^n$ with $\boldsymbol{\alpha} \in (\mathbb{Z}^n)^{\times d}$. Denote by $\mathrm{WV}_{\boldsymbol{\lambda}} V$ and $\mathrm{HWV}_{\boldsymbol{\lambda}} V$ the weight and the highest weight subspace of V of weight $\boldsymbol{\lambda}$, for some d-tuple of partitions.

2.1.4 Representation theory of the groups

For partition $\lambda \vdash M$ the bijection $Y : [M] \to D(\lambda)$ is called *Young tableau* of shape λ. Let $\mathrm{YT}(\lambda)$ be the set of all Young tableaux of shape λ. In other words, Y is the filling of a diagram with numbers $1, \ldots, M$. By $Y_{i,j} := Y^{-1}(i,j)$ denote the number written in a box (i,j).

Tableau Y is called *standard Young tableau* whenever $Y_{i,j} < \min(Y_{i+1,j}, Y_{i,j+1})$ (set $Y_{i,j} = \infty$ whenever $(i,j) \notin D(\lambda)$). Set of all standard Young tableaux denoted as $\mathrm{SYT}(\lambda)$.

The irreducible representation of the group $\mathrm{GL}(m)$ are indexed with partitions, known as *Weyl modules* and denoted as $V(\lambda)$. It is known, that weight space of weight λ is 1-dimensional in $V(\lambda)$ and called *highest weight line*. This line characterizes the irreducible representation.

Irreducible representations of symmetric group S_M are indexed by λ, denoted as $[\lambda]$ and referred to as *Specht modules*.

Recall the classical construction of irreducible representations of a symmetric group S_M. The group S_M acts on tableaux of shape λ by $(\pi T)_{i,j} = \pi(T_{i,j})$, also acts on tensor space $\bigotimes^M \mathbb{C}^n$ by permuting the tensor factors. Let $\lambda \vdash M$ be partition and $T \in \mathrm{SYT}(\lambda)$. Let $R(T)$ and $C(T)$ to be row and column stabilizers, then define

$$a_T = \sum_{g \in R(T)} g, \qquad b_T = \sum_{g \in R(T)} \mathrm{sgn}(g) g, \qquad c_T := b_T a_T.$$

It is known that c_T is a projector and

$$c_T : \bigotimes^M \mathbb{C}^m \to V(\lambda)_T, \qquad c_T : \mathbb{C}[S_M] \to [\lambda]_T$$

by right multiplication, where subscript T of $V(\lambda)_T$ indicates concrete irreducible as a subspace of $\bigotimes^m \mathbb{C}^n$.

2.1.5 Symmetrization and alternation

Let V be vector space. Consider tensor product space $\bigotimes^m V$. This space has two important subspaces.

a) Symmetric space

Define a map $\mathrm{Sym} : \bigotimes V \to \bigotimes V$ as follows:

$$\mathrm{Sym}(v_1 \otimes \ldots \otimes v_m) := \sum_{\omega \in S_m} v_{\omega(1)} \otimes \ldots \otimes v_{\omega(m)}.$$

and extended linearly. Then the operator Sym is a projector and its image, denoted $\mathcal{S}^m(V) := \mathrm{Sym}(\bigotimes^m V)$ for each $m \geqslant 0$, is a so called symmetric tensor power. This map can be used to define symmetric product on tensors:

$$T_1 \cdot \ldots \cdot T_m := \mathrm{Sym}(T_1 \otimes \ldots \otimes T_m)$$

where $T_1, \ldots, T_m \in T(V)$ can be arbitrary elements of a tensor algebra of V. This product makes $\bigoplus_{m \geqslant 0} S^m(V)$ an algebra called symmetric algebra which is isomorphic to the ring of polynomials of $\dim V$ variables, i.e.

$$\mathbb{C}[V]_m \cong \mathcal{S}^m(V)$$

b) Alternating space

Let the map Alt on $\bigotimes^m V$

$$\mathrm{Alt} : \bigotimes^m V \to \bigwedge^m V$$

be defined on simple tensors $v_1 \otimes \ldots \otimes v_m$, where $v_i \in V$, as

$$\mathrm{Alt}(v_1 \otimes \ldots \otimes v_m) = v_1 \wedge \ldots \wedge v_m := \sum_{\omega \in S_m} \mathrm{sgn}(\omega) v_{\omega(1)} \otimes \ldots \otimes v_{\omega(m)}$$

and denote its image $\bigwedge^m V := \mathrm{Alt}(\bigotimes^m V)$. The latter space is called skew-symmetric subspace or wedge algebra.

For $T \in A(\boldsymbol{\lambda})$ write

$$\bigotimes e_T = \bigotimes_{i=1}^{m} e_{T(i)}, \quad \prod e_T = \prod_{i=1}^{m} e_{T(i)}, \quad \bigwedge e_T = \bigwedge_{i=1}^{m} e_{T(i)},$$

where $T(i) \in \mathbb{N}^d$ and $e_{T(i)} = e_{T(i)_1} \otimes \ldots \otimes e_{T(i)_d}$.

2.1.6 Kronecker coefficients

For partitions λ, μ, ν of the same size n, the *Kronecker coefficient* $g(\lambda, \mu, \nu)$ is defined as the multiplicity of $[\nu]$ in the tensor product decomposition $[\lambda] \otimes [\mu]$, where $[\alpha]$ denotes the irreducible representation of S_n indexed by partition α. Here is the list of some known properties of Kronecker coefficients which will be useful for us.

Lemma 2.1.1. *The following properties hold:*
(a) (S_3 symmetry) $g(\lambda, \mu, \nu) = g(\lambda, \nu, \mu) = \cdots$ is invariant under permutations of λ, μ, ν.
(b) (Conjugation) $g(\lambda, \mu, \nu) = g(\lambda, \mu', \nu')$.
(c) (Trivial character) $g(1 \times k, \lambda, \mu) = \delta_{\lambda,\mu}$, $g(k \times 1, \lambda, \mu) = \delta_{\lambda,\mu'}$.
(d) (Square positivity [35]) $g(k \times k, k \times k, k \times k) > 0$ for every $k \in \mathbb{N}$.
(e) (Rectangular identities [36], [37, Cor. 4.4.14-15]) Let $a, b, c \in \mathbb{N}$.
If λ, μ, ν satisfy $\ell(\lambda) \leqslant a, \ell(\mu) \leqslant b, \ell(\nu) \leqslant ab$ then

$$g(\lambda, \mu, \nu) = g(\lambda + a \times bc, \mu + b \times ac, \nu + ab \times c).$$

If λ, μ, ν satisfy $\lambda \subseteq bc \times a, \mu \subseteq ac \times b, \nu \subseteq ab \times c$ then

$$g(\lambda, \mu, \nu) = g(bc \times a -_{bc} \lambda, ac \times b -_{ac} \mu, ab \times c -_{ab} \nu).$$

(f) (Size bounds [38]) $\max\{\lambda_1 : g(\lambda, \mu, \nu) > 0\} = |\mu \cap \nu|$. In particular, if $g(\lambda, \mu, \nu) > 0$ then $\ell(\lambda) \leqslant \ell(\mu)\ell(\nu)$.
(g) (Semigroup [39]) If $g(\lambda, \mu, \nu) > 0$ and $g(\alpha, \beta, \gamma) > 0$ then $g(\lambda + \alpha, \mu + \beta, \nu + \gamma) > 0$.

For partitions $\lambda^{(1)}, \ldots, \lambda^{(d)}$ of size n, the generalized Kronecker coefficient $g(\lambda^{(1)}, \ldots, \lambda^{(d)})$ is defined as the multiplicity of $[\lambda^{(d)}]$ in the tensor product decomposition $[\lambda^{(1)}] \otimes \cdots \otimes [\lambda^{(d-1)}]$.

2.2 Tensor invariants

In this section, the derivation of invariant polynomials within the framework of Schur-Weyl duality is considered. Initially, attension is focused on the scenario where the group $\mathrm{GL}(n)$ acts on the tensor space $\otimes^n(\mathbb{C}^n)$. This framework is then extended to include the space $\otimes^m(\mathbb{C}^n)^{\otimes d}$, with the corresponding group action facilitated by $\mathrm{GL}(n)^{\times d}$. Within this expanded context, a basis for the highest weight space associated with any given weight is derived, ultimately leading to the establishment of a basis for invariant non-commutative polynomials. Finally, these results are projected onto the symmetric space, thereby identifying a generating set of invariant polynomials. Throughout this exploration, several intrinsic properties of these polynomials are elaborated, highlighting their mathematical significance and applicative potential.

Let us first define several important definitions.

Definition 2.2.1 (Tables and maps)**.** For sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ with $|\alpha| = m$ let $A(\alpha)$ denote the set of permutations of a word $1^{\alpha_1} \ldots \ell^{\alpha_\ell}$ of length m. We say words from $A(\alpha)$ has weight α. We call equal letters of a word from $A(\alpha)$ as *blocks.*

A word $w \in A(\alpha)$ is said to be *lattice*, whenever in each prefix i at least as many times as $i+1$ for any positive i. Weight of a lattice word is always a partition, and the subset of lattice words of weight α is denoted as $A^+(\alpha) \subseteq A(\alpha)$.

For a d-tuple $\boldsymbol{\alpha} = (\alpha^{(1)}, \ldots, \alpha^{(d)})$ with $|\alpha^{(i)}| = m$ let $A(\boldsymbol{\alpha}) = A(\alpha^{(1)}) \times \ldots \times A(\alpha^{(d)})$ be the set of d-tuples of words. Throughout the text, a table $A \in A(\boldsymbol{\alpha})$ is interpreted in several ways:

- A is the d-tuple of words $(a^{(1)}, \ldots, a^{(d)})$, $a^{(i)} \in A(\alpha^{(i)})$;
- A is the $d \times m$ table with rows $A_1, \ldots, A_d$ and i-row equal to $a^{(i)}$;
- A is the map $[m] \to \mathbb{N}^d$ with $A(j) = (a_j^{(1)}, \ldots, a_j^{(d)})$, i.e. j-th column of a table.

Conventionally, we write A_i for i-th row and $A(j)$ for j-th column of a table A. Also, let $B(\boldsymbol{\alpha}) \subseteq A(\boldsymbol{\alpha})$ be the subset of tables *without repeating columns.* Analogically $B^+(\boldsymbol{\alpha}) \subseteq A^+(\boldsymbol{\alpha})$ denotes subset of tables with rows as lattice words and columns without repetitions.

Definition 2.2.2 (Signature function)**.** Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash m$ be a partition and $s \in A(\lambda)$. By $P_i = s^{-1}(i)$, denote the set of positions of letter i, referred to as *blocks.* For a map $\sigma : [m] \to \mathbb{N}$, define the following signature function:

$$\mathrm{sgn}_s(\sigma) := \mathrm{sgn}(\sigma(P_1)) \cdots \mathrm{sgn}(\sigma(P_\ell)) \tag{2.1}$$

where for a set $S = \{s_1 < \ldots < s_k\}$, denote $\sigma(S) = (\sigma(s_1), \ldots, \sigma(s_k))$ and

$$\operatorname{sgn}(p_1, \ldots, p_k) = \begin{cases} \operatorname{sgn}(p), & \text{if } p \text{ is a permutation of [k]} \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

For example, let $s = 112231132$, then

$$\begin{aligned} \operatorname{sgn}_s(121213423) &= \operatorname{sgn}(1234)\operatorname{sgn}(12)\operatorname{sgn}(12) = +1, \\ \operatorname{sgn}_s(233211421) &= \operatorname{sgn}(2314)\operatorname{sgn}(321)\operatorname{sgn}(12) = -1, \\ \operatorname{sgn}_s(243511421) &= \underbrace{\operatorname{sgn}(2414)\operatorname{sgn}(351)}_{=0 \text{ (not a permutation)}}\operatorname{sgn}(12) = 0. \end{aligned}$$

In other words, $\operatorname{sgn}_s(w)$ nulls whenever the word w does not have permutations at blocks of s of corresponding size; otherwise, the value equals the product of block permutation signatures. It is easy to see that only words of weight λ' can be non-zero at $\operatorname{sgn}_s(\cdot)$ for $s \in A(\lambda)$.

2.2.1 Invariants in Schur-Weyl duality, $d = 1$

First, a spanning set of $(\otimes^m \mathbb{C}^n)^{\mathrm{SL}(n)}$ for $m = nk$ is obtained. It is known that $(\otimes^m \mathbb{C}^n)^{\mathrm{SL}(n)} = 0$ whenever m is not divisible by n.

Definition 2.2.3. Define the ℓ-tensor $\det_\ell \in \bigotimes^\ell \mathbb{C}^n$ for $\ell \leqslant n$ as the (dual of) determinant:

$$\det_\ell := \sum_{\pi \in S_\ell} \operatorname{sgn}(\pi) e_{\pi(1)} \otimes \ldots \otimes e_{\pi(\ell)}, \tag{2.3}$$

i.e., the dual $\det^*(v_1, \ldots, v_\ell)$ is the determinant of the top $\ell \times \ell$ submatrix of an $n \times \ell$ matrix with vectors v_i interpreted as columns. It is not hard to verify that $\det_\ell$ is the highest weight vector of weight $(1, \ldots, 1, 0, 0, \ldots)$ with ℓ ones and $n - \ell$ zeroes. For any $\lambda \vdash m$ with $\ell(\lambda) \leqslant n$ and $\pi \in S_m$, define vectors in $\bigotimes^m \mathbb{C}^n$:

$$p_{\lambda,\mathrm{id}} := \det_{\lambda'_1} \otimes \ldots \otimes \det_{\lambda'_{\lambda_1}}, \quad p_{\lambda,\pi} = \pi \circ p_{\lambda,\mathrm{id}}. \tag{2.4}$$

Proposition 2.2.4. *For any $\lambda \vdash m$ and $\pi \in S_m$, the vectors $p_{\lambda,\pi}$ are the highest weight vectors of weight λ.*

These forms are now slightly modified for specific needs. Note that $p_{\lambda,\pi}$ and $p_{\lambda,\sigma}$ differ by ± 1 whenever permutations π and σ belong to the same coset of the Young subgroup $Y_{\lambda'}$. Cosets of $Y_{\lambda'}$ are indexed with words $A(\lambda')$.

Definition 2.2.5. For $t \in A(\lambda')$, define $p_t := p_{\lambda,\pi}$ where π is a lexicographically minimal representative of a coset t, i.e., π does not change the relative order of each block. For instance, $p_{(221),\text{id}} = \det_3 \otimes \det_2 = p_{11122}$ and $p_{(221),(35)} = -p_{11\mathbf{221}}$, here the minus sign appears due to transposition within block 2.

Roughly speaking, equal letters of t (block) represent positions where det is taken from. Then the concrete form of p_t can be derived:

$$p_t = \sum_{s \in A(\lambda)} \operatorname{sgn}_t(s) \cdot \bigotimes_{i=1}^{m} e_{s(i)} \in \mathrm{HWV}_\lambda \bigotimes^{m} \mathbb{C}^n.$$

Indeed, for a word $s \in A(\lambda)$ of weight λ, the coefficient of the weight vector $\otimes e_s$ in p_t is calculated as follows. Each block of t 'expects' a permutation of block size in corresponding positions of s, and the product of permutations from each block is exactly $\operatorname{sgn}_t(s)$.

For tableaux $T \in \mathrm{YT}(\lambda)$, let $col(T) \in A(\lambda')$ be a word $w_1 \dots w_m$ with w_i equal to the index of a column of T containing i, and $row(T) \in A(\lambda)$ similarly for rows. Then the maps $row : \mathrm{YT}(\lambda) \to A(\lambda)$ and $row : \mathrm{YT}(\lambda) \to A(\lambda')$ are indeed bijections. Note that $col(T)$ and $row(T)$ are *lattice words* whenever $T \in \mathrm{SYT}(\lambda)$. Define $p_T := p_{col(T)}$.

For instance, for tableau T in Figure 2.1, $col(T) = 1121323$ and $row(T) = 1213122$. As a result, $p_T = p_{1121323}$.

$$T = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 6 & 7 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

Figure 2.1 – Example of Young tableau

Theorem 2.2.6. *The set* $\{p_{col(T)}\}_{T \in \mathrm{SYT}(\lambda)}$ *forms the basis of* $\mathrm{HWV}_\lambda \bigotimes^m \mathbb{C}^n$.

Proof. By Schur-Weyl duality, there is a decomposition with respect to the action of $S_m \times \mathrm{GL}(n)$:

$$(\mathbb{C}^n)^{\otimes m} = \bigoplus_{\lambda : \lambda \vdash M, \ell(\lambda) \leqslant m} [\lambda] \otimes V(\lambda).$$

Moreover, inducing the action to $\mathrm{GL}(n)$, for $T \in \mathrm{SYT}(\lambda)$ let $V(\lambda)_T := c_T((\mathbb{C}^n)^{\otimes m})$ be the image of the Young symmetrizer, which is a $\mathrm{GL}(n)$-irreducible representation. By the theorem of the highest weight, each irreducible $V(\lambda)_T$ contains a unique highest weight line L_T. On the other hand,

$$\mathrm{HWV}_\lambda (\mathbb{C}^n)^{\otimes m} = \mathrm{HWV}_\lambda \bigoplus_{T \in \mathrm{SYT}(\lambda)} V(\lambda)_T = \bigoplus_{T \in \mathrm{SYT}(\lambda)} L_T.$$

In particular, the highest weight space is an irreducible S_m representation indexed by λ. It remains to show that $L_T = p_{col(T)}$. Set $W_\lambda = \mathrm{WV}_\lambda \bigotimes^m \mathbb{C}^n$. It is claimed that $c_T(W_\lambda) = L_T$. This can be seen from the diagram:

$$\begin{array}{ccc} W_\lambda & \subseteq & (\mathbb{C}^m)^{\otimes M} \\ {\scriptstyle c_T}\big\downarrow & & \big\downarrow{\scriptstyle c_T} \\ L_T & \subseteq & V(\lambda)_T \end{array}$$

and the fact that c_T preserves weight. Since $c_T(V^{\otimes M}) = V(\lambda)_T$ and $V(\lambda)_T \cap W_\lambda = L_T$, it follows that $c_T(W_\lambda) = L_T$.

Further, let $e_{row(T)} = e_{row(T)_1} \otimes \ldots \otimes e_{row(T)_m} \in W_\lambda$. Consider the product $c_T \cdot e_{row(T)} = b_T \cdot (a_T \cdot e_{row(T)})$. Note that for any $\pi \in R(T)$ (row stabilizer), $\pi e_{row(T)} = e_{row(T)}$, since positions from the fixed row of T are filled with the same letter in $row(T)$. Thus $a_T(e_T) = |R(T)| \cdot e_T$. Further, it turns out that $b_T(e_{row(T)}) = p_{col(T)}$. Indeed, b_T alternates positions from the same column, thus i-th instance of each letter in $row(T)$ corresponds to positions alternated in the i-th column. Hence, $p_{col(T)}$ is the highest weight line of $V(\lambda)_T$ and forms the basis for the highest weight space. □

2.2.2 General tensor invariants

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(d)})$ with each $\lambda^{(i)} \vdash m$ and $\ell(\lambda^{(1)}) \leqslant n$. Further, the basis of $\mathrm{HWV}_{\boldsymbol{\lambda}} \bigotimes^m (\mathbb{C}^n)^{\otimes d}$ is constructed. The key observation is that there is a natural isomorphism

$$\bigotimes^m \bigotimes^d \mathbb{C}^n \cong \bigotimes^d \bigotimes^m \mathbb{C}^n,$$

just by changing the order of the tensor product. The action of the groups S_m and $\mathrm{GL}(n)^{\times d}$ can be described by the table view of the tensor product represented in Figure 2.2, where S_m acts by permuting columns and each $\mathrm{GL}(n)$

	1		2		...		m	
$\mathrm{GL}(n)_1 \circlearrowleft$	$\mathbb{C}^n$	$\otimes$	$\mathbb{C}^n$	$\otimes$	...	$\otimes$	$\mathbb{C}^n$	$\otimes$
...	...	...	...	...	...	...	...	$\otimes$
$\mathrm{GL}(n)_d \circlearrowleft$	$\mathbb{C}^n$	$\otimes$	$\mathbb{C}^n$	$\otimes$	...	$\otimes$	$\mathbb{C}^n$	

Figure 2.2 – Diagrammatic representation of group action.

acts diagonally on the corresponding row.

Definition 2.2.7. For a d-tuple of partitions $\boldsymbol{\mu}$, define $A(\boldsymbol{\mu}) = A(\mu^{(1)}) \times \ldots \times A(\mu^{(d)})$ to be the set of d-tuples of words. An element $A = (a_1, \ldots, a_d) \in A(\boldsymbol{\mu})$

is represented as a $d \times m$ table with row i equal to a_i; $A(j) \in \mathbb{N}^d$ denotes the j-th column of the table A and A_i denotes the i-th row of the table. For another word $B = (b_1, \ldots, b_d) \in A(\boldsymbol{\nu})$, define $\mathrm{sgn}_A(B) = \mathrm{sgn}_{a_1}(b_1) \cdots \mathrm{sgn}_{a_d}(b_d)$.

Let $A = (a_1, \ldots, a_d) \in A(\boldsymbol{\lambda'})$ be a $d \times m$ table with rows a_i. Define a vector

$$P'_A := p_{a_1} \otimes \ldots \otimes p_{a_d} \in \mathrm{HWV}_{\boldsymbol{\lambda}} \bigotimes^d \bigotimes^m \mathbb{C}^m$$

and let P_A denote its image in $\bigotimes^m, \bigotimes^d \mathbb{C}^n$ under the isomorphism described above.

For a d-tuple of tableaux $T = (T_1, \ldots, T_d) \in \mathrm{YT}(\boldsymbol{\lambda})$, define $P'_T := P'_{col(T)} = p_{col(T_1)} \otimes \ldots \otimes p_{col(T_d)}$. Let $t_i := col(T_i) \in A((\lambda^{(i)})')$. The concrete form of P_T can be written as:

$$\begin{aligned}
P_T &= p_{t_1} \otimes \ldots \otimes p_{t_d} \\
&= \left(\sum_{s_1 \in A(\lambda^{(1)})} \mathrm{sgn}_{t_1}(s_1) \bigotimes_{i=1}^m e_{s_1(i)} \right) \otimes \ldots \otimes \left(\sum_{s_d \in A(\lambda^{(d)})} \mathrm{sgn}_{t_d}(s_d) \bigotimes_{i=1}^m e_{s_d(i)} \right) \\
&= \sum_{(s_1,\ldots,s_d) \in A(\boldsymbol{\lambda})} \mathrm{sgn}_{t_1}(s_1) \cdots \mathrm{sgn}_{t_d}(s_d) \bigotimes_{i=1}^m e_{s_1(i)} \otimes \ldots \otimes \bigotimes_{i=1}^m e_{s_d(i)} \\
&\cong \sum_{(s_1,\ldots,s_d) \in A(\boldsymbol{\lambda})} \mathrm{sgn}_{t_1}(s_1) \cdots \mathrm{sgn}_{t_d}(s_d) \bigotimes_{i=1}^m e_{s_1(i)} \otimes \ldots \otimes e_{s_d(i)},
\end{aligned}$$

hence, viewing $(s_1, \ldots, s_d) = S \in A(\boldsymbol{\lambda})$, rewrite

$$P_T = \sum_{S \in A(\boldsymbol{\lambda})} \mathrm{sgn}_T(S) \bigotimes_{i=1}^m e_{S(i)} \tag{2.5}$$

where $\mathrm{sgn}_T(S) = \mathrm{sgn}_{t_1}(S_1) \ldots \mathrm{sgn}_{t_d}(S_d)$.

Theorem 2.2.8. *The set* $\{P_{col(T)}\}_{T \in \mathrm{SYT}(\boldsymbol{\lambda})}$ *forms the basis of* $\mathrm{HWV}_{\boldsymbol{\lambda}} \bigotimes^m (\mathbb{C}^n)^{\otimes d}$.

Proof. Rearrange the tensor product

$$\mathrm{HWV}_{\boldsymbol{\lambda}}^{\mathrm{GL}(n)^{\times d}} \bigotimes^m \bigotimes^d \mathbb{C}^n \cong \bigotimes_{i=1}^d \mathrm{HWV}_{\lambda^{(i)}}^{\mathrm{GL}(n)} \bigotimes^m (\mathbb{C}^n)^{\otimes d}.$$

Thus, by Theorem 2.2.6, the tensors $P_{col(T)}$ for $T \in \mathrm{SYT}(\boldsymbol{\lambda})$ form the basis of the desired space. □

The above theorem implies the following statement. Recall that $A^+(\lambda)$ denotes the set of lattice words of weight λ and $A^+(\boldsymbol{\lambda}) = A^+(\lambda^{(1)}) \times \ldots \times A^+(\lambda^{(d)})$.

Corollary 2.2.9. *The set* $\{P_T\}_{T \in A^+(\boldsymbol{\lambda'})}$ *forms the basis of* $\mathrm{HWV}_{\boldsymbol{\lambda}} \bigotimes^m (\mathbb{C}^n)^{\otimes d}$.

Remark 2.2.10. This section considers only highest weight tensors over a cubical space $V_{cube} = (\mathbb{C}^n)^{\otimes d}$ rather than a generalized space $V_{general} = \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_d}$. However, it is easy to see that

$$\mathrm{HWV}_{\boldsymbol{\lambda}} \bigotimes^{m} V_{general} = \mathrm{HWV}_{\boldsymbol{\lambda}} \bigotimes^{m} V_{cube}$$

as long as $\ell(\lambda^{(i)}) \leqslant n_i$ and each $n_i \leqslant n$. If $\ell(\lambda^{(i)}) > n_i$, then the l.h.s. space nulls and the r.h.s. space may not null. So it is enough to consider the cubical case.

2.2.3 Conclusion

In this chapter, the theoretical foundations of tensor invariants were explored, focusing on the intricate structures of tensors and hypermatrices, as well as their symmetries and representation theories. By investigating tensor product spaces and their associated highest weight vectors, a comprehensive understanding of the mathematical framework underlying these concepts was developed. The study of partitions, highest weight vectors, and their representations provided a solid groundwork for examining tensor invariants in more complex scenarios.

Particularly, Schur-Weyl duality was employed to derive invariant polynomials, extending the analysis from simple tensor spaces to higher-dimensional ones, such as $\otimes^m(\mathbb{C}^n)^{\otimes d}$. This approach facilitated the identification of bases for highest weight spaces, leading to the establishment of a basis for invariant non-commutative polynomials. Furthermore, these results were projected onto symmetric spaces, resulting in the identification of a generating set of invariant polynomials and an understanding of their intrinsic properties.

2.3 Invariant polynomials

In this section, attention is concentrated on the derivation of $G = \mathrm{SL}(n_1) \times \ldots \times \mathrm{SL}(n_d)$-invariant polynomials over tensor space $V_{general} := \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_d}$. Note that these polynomials are included in and $\mathrm{Sym}^m(V) = \mathcal{S}(\bigotimes^m V)$ where $\mathcal{S}$ is a projector to symmetric subspace (symmetrizer). Denote

$$\mathrm{Inv}_d(n_1, \ldots, n_d) := \mathbb{C}[\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_d}]^{\mathrm{SL}(n_1)\times\ldots\times\mathrm{SL}(n_d)}$$

the ring of G-invariant polynomials on V which is graded

$$\mathbb{C}[V_{general}]^G = \bigoplus_{m \geqslant 0} \mathbb{C}[V_{general}]^G_m,$$

where $\mathbb{C}[V_{general}]^G_m$ is the space of G-invariant homogeneous polynomials of degree m denoted by

$$\mathrm{Inv}(n_1, \ldots, n_d)_m := \mathbb{C}[V_{general}]^G_m.$$

The following fact is well known, e.g., [40].

Proposition 2.3.1. *Let $k_i = m/n_i$. It holds that*

$$\dim \mathrm{Inv}(n_1, \ldots, n_d)_m = g(n_1 \times k_d, \ldots, n_d \times k_d).$$

In the case $n_1 = \ldots = n_d = n$, use the notation

$$\mathrm{Inv}_d(n) := \mathrm{Inv}(n, \ldots, n) \text{ and } \mathrm{Inv}_d(n)_m := \mathrm{Inv}(n, \ldots, n)_m.$$

It is known that if the space $\mathrm{Inv}(n_1, \ldots, n_d)_m$ is nonzero then n_i divides m. A *fundamental invariant* is a polynomial in $\mathbb{C}[V]^G$ of the smallest positive degree.[1] Let

$$\delta_d(n) := \min\{m : \dim \mathrm{Inv}_d(n)_m > 0\} \in n\mathbb{N}$$

be the degree of a fundamental invariant which is known to be a multiple of n, cf. Proposition 2.3.5. When d is even, there is a unique (up to a scalar) fundamental invariant of degree n, originally due to Cayley [16]; it is a straightforward generalization of the determinant, see Example 2.3.8. But when d is odd, the situation turns out to be much more complicated; in particular, the degrees and descriptions of fundamental invariants are not known in general. The main goal of this section is to introduce the methodology of generating such (fundamental) invariant polynomials, as well as show the following bounds.

Theorem 2.3.2. *Let $d \geqslant 3$ be odd. The following properties hold:*

[1] Terminology as in [41]; more broadly, fundamental invariants may also refer to ring generators, e.g. [42], but here the smallest invariants are meant.

(i) The bounds

$$n\lceil n^{1/(d-1)}\rceil \leqslant \delta_d(n) \leqslant n^2.$$

(ii) The lower bound is achieved at least in the following cases

$$\delta_d(n) = n\lceil n^{1/(d-1)}\rceil \text{ for } n \in \{1, \dots, 2^{d-1}\} \cup \{3^{d-1}, \dots, 4^{d-1}\} \cup \{k^{d-1} - 1, k^{d-1} : k \in \mathbb{N}_{\geqslant 2}\},$$

$n \in [k^{d-1} - \sqrt{k}/2 + 1, k^{d-1}]$ if k is even, and also $\delta_d(n) \in \{3n, 4n\}$ for $n \in \{2^{d-1} + 1, \dots, 3^{d-1}\}$.

Consider the following four statements:
$A_{d,k}$: $F_{d,k}(I_{k^{d-1}}) \neq 0$. ($d$-dimensional Alon–Tarsi)
$B_{d,k}$: $g_d(n, k) > 0$ for all $n \leqslant k^{d-1}$. (rectangular Kronecker positivity)
$C_{d,k}$: $\delta_d(n) = nk$ for all $n \in [(k-1)^{d-1}, k^{d-1}]$. (degrees of invariants)
$C'_{d,k}$: $\delta_d(n) \in \{nk, n(k+1)\}$ for all $(k-1)^{d-1} < n \leqslant k^{d-1}$.
These statements are related. Namely, for every odd $d \geqslant 3$ the implication diagram in Figure 2.3 represents the relations.

The statement $A_{d,k}$ (which is false for odd k, cf. Corollary 2.5.13) is a d-dimensional analogue of the celebrated *Alon–Tarsi conjecture* on Latin squares [43]. Hence, conditionally on $A_{3,k}$ for even k, the positivity of Kronecker coefficients $B_{d,k}$ is obtained and then degree values $C_{d,k}$ and $C'_{d,k-1}$ for all odd $d \geqslant 3$. It is shown that $A_{d,2}$ is true for all $d \geqslant 2$, see Proposition 2.5.15; in [41] it is noted that $A_{3,k}$ is true for $k = 2, 4$. Computations that $B_{3,k}$ is true for $k = 2, 4$ are also provided, see Appendix B. As a result, the true statements $C_{d,2}$, $C_{d,4}$, $C'_{d,3}$ for all odd $d \geqslant 3$ are reflected in Theorem 2.3.2 (ii).

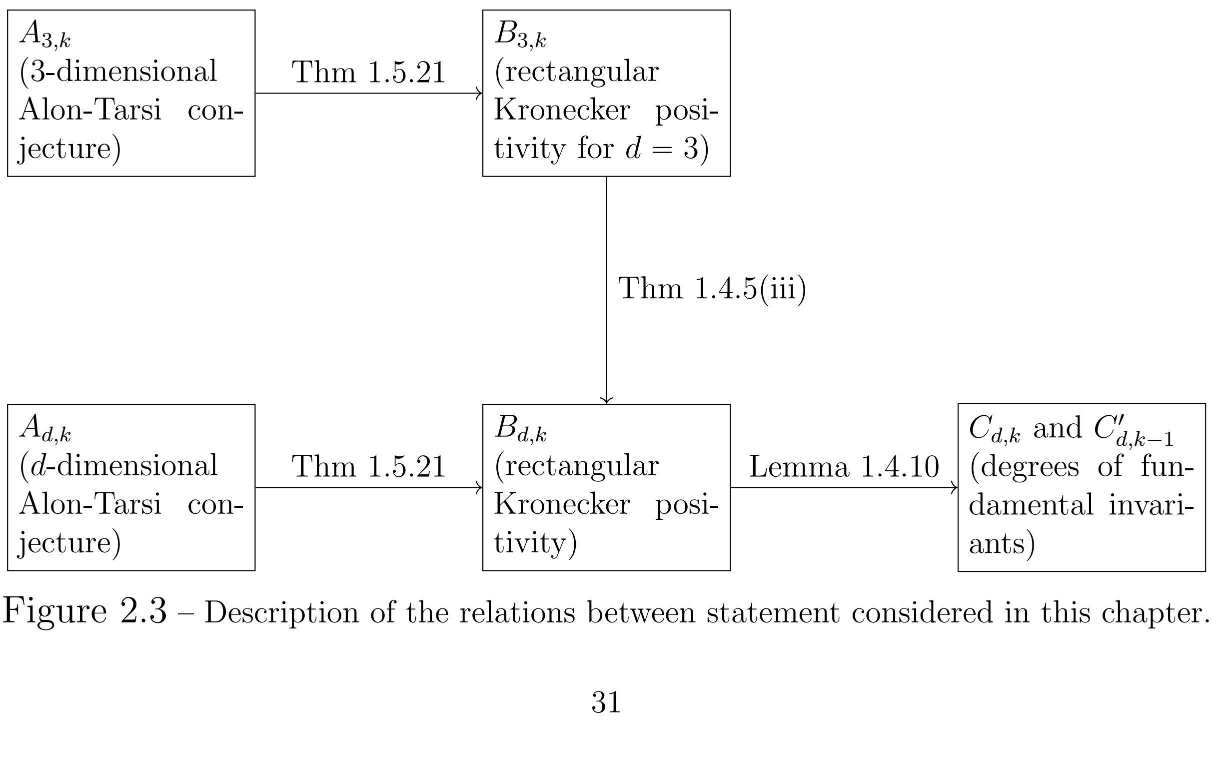

Figure 2.3 – Description of the relations between statement considered in this chapter.

It also seems that $B_{d,k}$ holds for all odd $d \geqslant 5$ and all k. Regarding the degree sequence $\delta_d(n)$, it is conjectured more precisely the following.

Conjecture 2.3.3. *Let $d \geqslant 3$ be odd. Then $\delta_d(n) = n\lceil n^{1/(d-1)}\rceil$ with the only exceptional cases $d = 3$ and odd $n = k^2 - 2$ for which $\delta_d(n)$ is larger by n.*

The derivation of a generating set of invariant polynomials is first described, further showing their properties and null conditions. After that, by analyzing degree and dimension sequences, bounds on $\delta_d(n)$ are recovered and results concerning the d-dimensional Alon-Tarsi conjecture and Latin hypercubes are shown.

2.3.1 Generating set

The definition of a generating set is started from. These polynomials are G-invariant and describe the space of invariants as follows. It is recalled that for any vector space V

$$\mathbb{C}[V]_m \cong \mathrm{Sym}^m(V) \subset \bigotimes^m V$$

Definition 2.3.4. Let $X \in V$ be given as a hypermatrix and $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(d)}) \vdash m$. Define the polynomials $\{\Delta_T\}$ indexed by tables T with row words $T = (T_1, \ldots, T_d) \in A(\lambda^{(1)}, \ldots, \lambda^{(d)})$ as follows

$$\Delta_T(X) := \sum_{\sigma:[m]\to[n]^d} \mathrm{sgn}_T(\sigma) \prod_{i=1}^{M} X_{\sigma(i)} \in \mathrm{WV}_{\boldsymbol{\lambda}}\mathbb{C}[V_{general}]_m \tag{2.6}$$

where, as usual, $X_{(i_1,\ldots,i_d)} = \langle X, e_{i_1} \otimes \ldots \otimes e_{i_d}\rangle$.

An analogous statement can be found in [44, Prop. 3.10] and [45, Prop. 3.10].

Proposition 2.3.5. *For $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(d)}) \vdash m$ with $\ell(\lambda^{(i)}) \leqslant n_i$, the polynomials $\{\Delta_T\}$ indexed by d-tuples of tables $T \in A^+(\boldsymbol{\lambda})$ span the space $\mathrm{HWV}_{\boldsymbol{\lambda}}\mathbb{C}[V_{general}]_m$.*

In particular, for $T \in A^+(n_1 \times k_1) \times \ldots \times A^+(n_d \times k_d)$ polynomials span the space $\mathrm{Inv}(n_1, \ldots, n_d)_m$, for $m = n_i k_i$. If n_i does not divide m for some i, this space is empty.

Proof. Consider the highest weight tensor $P_T \in \mathrm{HWV}_{\boldsymbol{\lambda}} \bigotimes^m V$ (see Remark 2.2.10). The projection of P_T to $\mathbb{C}[V]_m$ is calculated:

$$\begin{aligned}\mathrm{Sym}^m(P_T) &= \sum_{\pi\in S_m} \pi \left(\sum_{\sigma:[m]\to[n_1]\times\ldots\times[n_d]} \mathrm{sgn}_T(\sigma) \bigotimes_{i=1}^m e_{\sigma(i)} \right) \\ &= \sum_{\sigma:[m]\to[n_1]\times\ldots\times[n_d]} \mathrm{sgn}_T(\sigma) \sum_{\pi\in S_m} \pi \left(\bigotimes_{i=1}^m e_{\sigma(i)} \right)\end{aligned}$$

$$= \sum_{\sigma:[m]\to[n_1]\times\ldots\times[n_d]} \operatorname{sgn}_T(\sigma) \prod_{i=1}^{m} e_{\sigma(i)} = \Delta_T,$$

which exactly matches (2.6). Now, the proof follows by Corollary 2.2.9 and the simple fact from linear algebra that the projection of a basis of general space spans the image of a projector. □

Remark 2.3.6. In [44] and [45, Ex. 7.18] these Δ polynomials are presented in a slightly different form; they are indexed by d-tuples of permutations of $[m]$ (i.e. if set partitions S are not ordered in increasing order initially). The versions differ up to a sign.

Remark 2.3.7. While the form of Δ polynomials is explicit, it can be difficult to understand some basic questions about them, such as when Δ_T is a nonzero polynomial or if its evaluation on unit tensor is nonzero.

2.3.2 Examples

Some examples of the invariant polynomials Δ are shown.

Example 2.3.8 (Cayley's first hyperdeterminant)**.** Historically the first and the most simple-looking fundamental invariant arises as follows. Let T be the $d \times n$ balanced table of all ones

$$T = \begin{pmatrix} 11\ldots1 \\ \ldots \\ 11\ldots1 \end{pmatrix}.$$

Then

$$\Delta_T(X) = \sum_{\sigma_1,\ldots,\sigma_d \in S_n} \operatorname{sgn}(\sigma_1 \cdots \sigma_d) \prod_{i=1}^{n} X_{\sigma_1(i),\sigma_2(i),\ldots,\sigma_d(i)}.$$

This function is nontrivial when d is even; otherwise, it is identically 0. This invariant was introduced by Cayley [16].

Example 2.3.9 (Cayley's second hyperdeterminant)**.** Let $d = 3$, $n = 2$ and consider the following balanced table:

$$T = \begin{pmatrix} 1122 \\ 1122 \\ 1212 \end{pmatrix}.$$

Then

$$-\frac{1}{2}\Delta_T(X) = X_{111}^2X_{222}^2 + X_{112}^2X_{221}^2 + X_{121}^2X_{212}^2 + X_{211}^2X_{122}^2$$

$$
\begin{aligned}
&- 2(X_{111}X_{112}X_{221}X_{222} + X_{111}X_{121}X_{212}X_{222} + X_{111}X_{211}X_{122}X_{222} \\
&+ X_{112}X_{121}X_{212}X_{221} + X_{112}X_{211}X_{122}X_{221} + X_{121}X_{211}X_{122}X_{212}) \\
&+ 4(X_{111}X_{122}X_{212}X_{221} + X_{112}X_{121}X_{211}X_{222}).
\end{aligned}
$$

This function is the unique (up to scalar) fundamental invariant of $\mathrm{Inv}(2,2,2)$ and in fact generates this ring [46, Theorem 13]. Originally, it was also computed by Cayley [17], and it is the simplest example of the *geometric hyperdeterminant* studied in [18].

Example 2.3.10 (2 × 2 × 3 geometric hyperdeterminant)**.** Let $d = 3$, $n_1 = 2$, $n_2 = 2$ and $n_3 = 3$ and consider the following balanced table:

$$
T = \begin{pmatrix} 112233 \\ 112233 \\ 121212 \end{pmatrix}.
$$

Then Δ_T is the unique (up to scalar) fundamental invariant of the ring $\mathrm{Inv}(2,2,3)$, and $-\frac{1}{12}\Delta_T$ gives the geometric hyperdeterminant. See [47] for related computations.

Example 2.3.11 ($3\times3\times3$ ring of invariants)**.** The ring $\mathrm{Inv}(3,3,3)$ is generated by three homogeneous polynomials of degrees $6, 9$ and 12, see [42]. As generators, one can use the invariant polynomials indexed by the following balanced tables:

$$
T_1 = \begin{pmatrix} 111222 \\ 112221 \\ 122211 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 111222333 \\ 112223331 \\ 122233311 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 111222333444 \\ 111222333444 \\ 123124134234 \end{pmatrix}.
$$

(To compare with generators in [42], $\Delta_{T_1}, \Delta_{T_2}$ differ up to scalar, and up to $\Delta_{T_3} + c\Delta_{T_1}^2$.) Note that the geometric hyperdeterminant of format $3 \times 3 \times 3$ has degree 36 and can be written as $P(\Delta_{T_1}, \Delta_{T_2}, \Delta_{T_3})$ for some polynomial P, see [48] for an explicit presentation. In the expansion of P, each monomial of type $\Delta_{T_1}^{\alpha}\Delta_{T_2}^{\beta}\Delta_{T_3}^{\gamma}$ can be written as a 3×36 balanced table: horizontally concatenate T_1 α times, T_2 β times, and T_3 γ times, so that each concatenated table is shifted in numbers (see § c) for this operation). This way the geometric hyperdeterminant can be written as a linear combination of the Δ polynomials.

2.3.3 Properties

This section presents some basic properties of the Δ polynomials.

a) Relative invariance

Proposition 2.3.12 (Relative GL-invariance)**.** *Let* $X \in V$*,* $(A_1, \dots, A_d) \in \mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_d)$*. For* $T \in A(n_1 \times k_1) \times \ldots \times A(n_d \times k_d)$ *we have*

$$\Delta_T((A_1, \dots, A_d) \cdot X) = \det(A_1)^{k_1} \cdots \det(A_d)^{k_d}\, \Delta_T(X).$$

Proof. Since $(A_1, \dots, A_d) \cdot X = ((A_1, \dots, I) \cdot \dots \cdot (I, \dots, A_d)) \cdot X$ it is enough to check the identity for an action of one component, say for $(A_1, I, \dots, I)$. Denote $A = A_1$ and $Y = (A, I, \dots, I) \cdot X$. Let $S = (S^1, \dots, S^d)$ be the d-tuple of set partitions, where S^i is the set partition of $[m]$ corresponding to blocks of a word T_i. Then

$$\begin{aligned}
\Delta_S(Y) &= \sum_{\sigma_1,\dots,\sigma_d} \mathrm{sgn}_{S^1}(\sigma_1) \cdots \mathrm{sgn}_{S^d}(\sigma_d) \prod_{i=1}^{m} \left(\sum_{\ell_i=1}^{n_1} A_{\sigma_1(i),\ell_i} X_{\ell_i,\sigma_2(i),\dots,\sigma_d(i)} \right) \\
&= \sum_{\ell:[m]\to[n_1]} \sum_{\sigma_1,\dots,\sigma_d} \mathrm{sgn}_{S^1}(\sigma_1) \cdots \mathrm{sgn}_{S^d}(\sigma_d) \prod_{i=1}^{m} A_{\sigma_1(i),\ell(i)} X_{\ell(i),\sigma_2(i),\dots,\sigma_d(i)} \\
&= \sum_{\ell:[m]\to[n_1]} \sum_{\sigma_2,\dots,\sigma_d} \mathrm{sgn}_{S^2}(\sigma_2) \cdots \mathrm{sgn}_{S^d}(\sigma_d) \prod_{i=1}^{m} X_{\ell(i),\sigma_2(i),\dots,\sigma_d(i)} \\
&\qquad \times \left(\sum_{\sigma_1} \mathrm{sgn}_{S^1}(\sigma_1) A_{\sigma_1(i),\ell(i)} \right)
\end{aligned}$$

where $\sigma_i : [m] \to [n_i]$ for $i \in [d]$. Note that σ_1 splits into $k_1 = m/n_1$ subpermutations according to S^1 when $\mathrm{sgn}_{S^1}(\sigma_1) \neq 0$; hence the last sum factors into k_1 determinants of matrices formed by the columns $\ell(1), \dots, \ell(m)$ according to S^1. Since permutation of columns alters the sign of the determinant by its sign and vanishes when two columns are equal, we have

$$\Delta_S(Y) = \det(A)^{k_1} \sum_{\ell,\sigma_2,\dots,\sigma_d} \mathrm{sgn}_{S^1}(\ell) \ldots \mathrm{sgn}_{S^d}(\sigma_d) \prod_{i=1}^{m} X_{\ell(i),\dots,\sigma_d(i)} = \det(A)^{k_1} \Delta_S(X)$$

and the claim follows. □

A hypermatrix X corresponding to a tensor in V has n_ℓ *parallel slices* in the direction $\ell \in [d]$ given by fixing the ℓ-th coordinate $i_\ell \in [n_\ell]$.

Corollary 2.3.13. *The polynomials* Δ_S *satisfy:*

1 *Exchanging any two parallel slices in direction* i *changes* Δ_S *by* $(-1)^{m/n_i}$*.*

2 Δ_S *is a homogeneous polynomial in the entries of each slice. The degree of homogeneity is the same for parallel slices in direction* i *and is equal to* m/n_i*.*

3 *Δ_S does not change if a scalar multiple of a parallel slice is added to any slice.*

4 *$\Delta_S = 0$ if there are two parallel slices proportional to each other.*

Remark 2.3.14. For even d and minimal degree $m = n_1 = \cdots = n_d$, such properties in fact characterize Cayley's first hyperdeterminant, see [49, Prop. 3.2].

b) Column swap

Let T be a balanced $d \times m$ table. For $i, j \in [m]$ denote by $(i, j)T$ the table resulting in the swap of the i-th and j-th columns of T.

Lemma 2.3.15. *Let $i < j \in [m]$. Then*

$$\Delta_T = (-1)^\ell \Delta_{(i,j)T}$$

where

$$\ell = \sum_{r=1}^{d} \#\{c \in [i, j] : T_{r,i} = T_{r,c}\} + \#\{c \in [i, j] : T_{r,j} = T_{r,c}\} - \delta_{T_{r,i}T_{r,j}}. \tag{2.7}$$

Proof. Each monomial in the expansion (2.6) of Δ_T can be bijectively mapped to the same monomial of $\Delta_{(i,j)T}$. For a tuple of maps $\sigma = (\sigma_1, \ldots, \sigma_d)$ as in the expansion (2.6), associate another tuple of maps $\tau = (\tau_1, \ldots, \tau_d)$ given by $\tau_r = \sigma_r \circ (i, j)$ (here $\circ$ is the composition) for each $r \in [d]$. Clearly, $\{\sigma(1), \ldots, \sigma(m)\} = \{\tau(1), \ldots, \tau(m)\}$ as sets of vectors. Denote

$$\ell_r = \#\{c \in (i, j) : T_{r,i} = T_{r,c}\} + \#\{c \in (i, j) : T_{r,j} = T_{r,c}\} - \delta_{T_{r,i}T_{r,j}} \tag{2.8}$$

and it follows that $\mathrm{sgn}_{T_r}(\sigma_r) = (-1)^{\ell_r} \mathrm{sgn}_{(i,j)\circ T_r}(\tau_r)$ where T_r is the r-th row of T. Let $x = T_{r,i}$ and $y = T_{r,j}$. Except for x-th and y-th sub-permutations, the relative order of other sub-permutations does not change, hence their contributions remain the same. If $x = y$ then the number of inversions of x-th sub-permutations in σ_r and τ_r differ by one, and ℓ_r is odd, as needed. If $x \neq y$ then the number of inversions in x-th (or y-th) sub-permutation changes by the number of elements of this permutation that lie in the range $[i, j]$ of T_r, which is exactly the terms of (2.8), as needed. Finally,

$$\mathrm{sgn}_T(\sigma) = \prod_{r=1}^{d} \mathrm{sgn}_{T_r}(\sigma_r) = \prod_{r=1}^{d} (-1)^{\ell_r} \mathrm{sgn}_{(i,j)\circ T_r}(\tau_r) = (-1)^\ell \mathrm{sgn}_{(i,j)T}(\tau)$$

and the proof follows. □

Corollary 2.3.16. *Let d be odd. If T has two equal columns then $\Delta_T = 0$.*

Proof. Suppose $i \neq j$ are indices of two equal columns in T. Then for each row

$m \in [d]$ we have $T_{m,i} = T_{m,j}$. Then we have $\Delta_T = (-1)^\ell \Delta_{(i,j)T} = (-1)^\ell \Delta_T$ and

$$\ell = \sum_{m=1}^{d} (2\#\{c \in [i,j] : T_{m,i} = T_{m,c}\} - 1)$$

which is an odd number. Hence $\Delta_T = -\Delta_T$ and the claim follows. □

Remark 2.3.17. These properties show that the order of columns of T can be disregarded and viewed as a set of its columns (as vectors) by sorting them in lexicographical order; this point of view will be especially useful in § 2.5.2.

Remark 2.3.18. Note that having distinct columns of T is a necessary condition for Δ_T to be nonzero but not sufficient. For example,

$$T = \begin{pmatrix} 112233 \\ 112233 \\ 122313 \end{pmatrix} \qquad \Delta_T = 0.$$

Note that in this case $d = 3, n = 2$ and there are no invariants of degree 6.

Remark 2.3.19. Swapping the rows i and j of T, denoted as $(i,j)_r T$, affects the polynomial Δ_T in the following way:

$$\Delta_{(i,j)_r T}(X) = \Delta_T(X^{t(i,j)})$$

where $X^{t(i,j)}$ is the result of transposing X along directions i and j. In the case of Cayley's first hyperdeterminant, $\Delta(X) = \Delta(X^{t(i,j)})$ (same as for ordinary determinant), but for general functions Δ this property does not hold. For example, sometimes a swap of two rows in a balanced table can be replaced with column swaps for odd ℓ and the result will differ by sign.

Remark 2.3.20. Note also that the invariant polynomials Δ from the same space can be equivalent up to a scale $\neq \pm 1$, which is different from previous instances of equivalences. For example, let $d = n = k = 3$ and consider the following balanced tables:

$$T_1 = \begin{pmatrix} 111222333 \\ 112123233 \\ 121\color{red}{2233}\color{black}{13} \end{pmatrix}, \quad T_2 = \begin{pmatrix} 111222333 \\ 112123233 \\ 121\color{red}{3223}\color{black}{13} \end{pmatrix}, \quad T_3 = \begin{pmatrix} 111222333 \\ 112123233 \\ 121\color{red}{3232}\color{black}{13} \end{pmatrix},$$

where the red entries indicate the difference between the tables. Then

$$\Delta_{T_1} = -4\Delta_{T_2} = 2\Delta_{T_3} \neq 0.$$

c) Horizontal concatenation

Let T_1 and T_2 be $d \times m_1$ and $d \times m_2$ balanced tables, respectively. Denote by (T_1T_2) the $d \times (m_1 + m_2)$ balanced obtained by the *horizontal concatenation* of T_1 and T_2 so that each entry of T_2 in row i is increased by m_1/n_i (to make the entries in T_1 and T_2 different). For example,

$$T_1 = \begin{pmatrix} 111222 \\ 112122 \\ 122112 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 111222333 \\ 112123233 \\ 121231323 \end{pmatrix}, \quad (T_1T_2) = \begin{pmatrix} 111222\mathbf{333444555} \\ 112122\mathbf{334345355} \\ 122112\mathbf{343453545} \end{pmatrix}$$

It is easy to see that the product of elements of the ring $\mathrm{Inv}(n_1, \dots, n_d)$ can be written via horizontal concatenations.

Proposition 2.3.21. *Let $\Delta_{T_1}, \Delta_{T_2} \in \mathrm{Inv}(n_1, \dots, n_d)$. Then*

$$\Delta_{T_1} \cdot \Delta_{T_2} = \Delta_{(T_1T_2)} \in \mathrm{Inv}(n_1, \dots, n_d).$$

d) Vertical concatenation

Let T_1 and T_2 be $\ell \times m$ and $(d - \ell) \times m$ balanced tables respectively so that $\Delta_{T_1} \in \mathrm{Inv}(n_1, \dots, n_\ell)$ and $\Delta_{T_2} \in \mathrm{Inv}(n_{\ell+1}, \dots, n_d)$. Denote by $T = \binom{T_1}{T_2}$ the $d \times M$ balanced table obtained by the *vertical concatenation* of T_1 and T_2. Note that $\Delta_T \in \mathrm{Inv}(n_1, \dots, n_d)$.

Proposition 2.3.22. *Let $Y \in V_1 = \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_\ell}$, $Z \in V_2 = \mathbb{C}^{n_{\ell+1}} \otimes \cdots \otimes \mathbb{C}^{n_d}$. We have*

$$\Delta_T(Y \otimes Z) = \Delta_{T_1}(Y) \cdot \Delta_{T_2}(Z),$$

where $Y \otimes Z \in V$ is the outer tensor product.

Proof. Let $X = Y \otimes Z$. We have

$$\begin{aligned} \Delta_T(X) &= \sum_{\sigma_1, \dots, \sigma_d} \mathrm{sgn}_T(\sigma) \prod_{i=1}^{M} X_{\sigma_1(i), \dots, \sigma_d(i)} \\ &= \sum_{\sigma_1, \dots, \sigma_d} \mathrm{sgn}_{T_1}(\sigma_A)\, \mathrm{sgn}_{T_2}(\sigma_B) \prod_{i=1}^{M} Y_{\sigma_A(i)} Z_{\sigma_B(i)} = \Delta_{T_1}(Y) \cdot \Delta_{T_2}(Z), \end{aligned}$$

where for a subset $A = \{a_1 < \ldots < a_k\} \subseteq [d]$ $\sigma_A = (\sigma_{a_1}, \dots, \sigma_{a_k})$ and $\sigma_A(i) = (\sigma_{a_1}(i), \dots, \sigma_{a_k}(i))$. ∎

Corollary 2.3.23. *Assume Δ_{T_1} is identically 0. Then the restriction of Δ_T on $V_1 \otimes V_2$ is identically 0.*

Corollary 2.3.24. *If Δ_{T_1}, Δ_{T_2} are nonzero polynomials, then $\Delta_{\binom{T_1}{T_2}}$ is also a non-zero polynomial.*

2.3.4 Conclusion

In this section, the derivation and properties of G-invariant polynomials over the tensor space $V_{general} := \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_d}$ were explored, focusing on their role within the context of symmetrized tensor products. By defining the ring of G-invariant polynomials and establishing their graded structure, a comprehensive framework for understanding these polynomials' fundamental properties was provided.

Through the introduction of generating sets and explicit examples, it was illustrated how specific balanced tables can be utilized to construct invariant polynomials, including Cayley's hyperdeterminants. Furthermore, key properties such as relative GL-invariance, column swaps, and the effects of horizontal and vertical concatenations on these polynomials were examined. These properties not only underscore the structural complexity of invariant polynomials but also reveal the intricate relationships between different configurations of balanced tables.

Several critical results were established, such as bounds on the degrees of fundamental invariants and the implications of various statements related to the d-dimensional Alon-Tarsi conjecture. These findings contribute to a deeper understanding of the algebraic and combinatorial aspects of invariant polynomials, paving the way for future research in this area. Through rigorous analysis and detailed examples, this section provides a solid foundation for further exploration of invariant polynomials in quantum information theory and beyond.

2.4 Degrees and dimensions

In this section the degree sequences and dimensions of invariant polynomials is analyzed.

2.4.1 Degree bounds

More generally, let us denote the smallest degrees of invariants as follows:

$$\delta(n_1, \ldots, n_d) := \min\{m : \dim \mathrm{Inv}(n_1, \ldots, n_d)_m > 0\}$$

so that $\delta_d(n) = \delta(n, \ldots, n)$ (d times). Note that n_i divides $\delta(n_1, \ldots, n_d)$ for all $i \in [d]$ and hence

$$\delta(n_1, \ldots, n_d) \geqslant \mathrm{lcm}(n_1, \ldots, n_d).$$

Below another lower bound is obtained, which sometimes can be larger than the last one. A kind of recursive upper bound is also shown.

Theorem 2.4.1. *The following bounds hold:*

(i) Let $d \geqslant 3$ be odd. Then

$$\delta(n_1, \ldots, n_d) \geqslant \lceil (n_1 \ldots n_d)^{1/(d-1)} \rceil.$$

(ii) Let $\ell \in [2, d-2]$ and $d > 3$. Then

$$\delta(n_1, \ldots, n_d) \leqslant \mathrm{lcm}(\delta(n_1, \ldots, n_\ell), \delta(n_{\ell+1}, \ldots, n_d)).$$

(Assuming all $\delta(\cdots) < \infty$.)

Proof. (i) Let $m = \delta(n_1, \ldots, n_d)$ and T be a balanced table associated with minimal degree m invariant polynomial $\Delta_T \neq 0$. Then each column of T is one of the possible $k_1 \cdots k_d$ columns, where $k_i = m/n_i$. Hence if $m > k_1 \ldots k_d$ then the table T has two equal columns and by Corollary 2.3.16 get $\Delta_T = 0$. Therefore,

$$m \leqslant k_1 \ldots k_d = m^d (n_1 \ldots n_d)^{-1}$$

which implies the inequality.

(ii) Denote $m_1 = \delta(n_1, \ldots, n_\ell), m_2 = \delta(n_{\ell+1}, \ldots, n_d)$ and let $M = \mathrm{lcm}(m_1, m_2)$. Construct a nonzero invariant polynomial in $\mathrm{Inv}(n_1, \ldots, n_d)$ using invariants from $\mathrm{Inv}(n_1, \ldots, n_\ell)$ and $\mathrm{Inv}(n_{\ell+1}, \ldots, n_d)$. Let $\Delta_{T_1} \in \mathrm{Inv}(n_1, \ldots, n_\ell)_{m_1}$ and $\Delta_{T_2} \in \mathrm{Inv}(n_{\ell+1}, \ldots, n_d)_{m_2}$ be fundamental (nonzero) invariants. Let $\tilde{T}_1 = (T_1 \cdots T_1)$ (M/m_1 times) and $\tilde{T}_2 = (T_2 \cdots T_2)$ (M/m_2 times) be $\ell \times M$ and $(d-\ell) \times M$ balanced tables obtained by horizontal concatenations. Let $T = \binom{\tilde{T}_1}{\tilde{T}_2}$ be $d \times M$ balanced table obtained by vertical concatenation. Let Y, Z be tensors such that $\Delta_{T_1}(Y) \neq 0, \Delta_{T_2}(Z) \neq 0$. By Propositions 2.3.22 and 2.3.21, follows

$$\Delta_T(Y \otimes Z) = \Delta_{\tilde{T}_1}(Y) \cdot \Delta_{\tilde{T}_2}(Z) = \Delta_{T_1}(Y)^{M/m_1} \cdot \Delta_{T_2}(Z)^{M/m_2} \neq 0.$$

Hence Δ_T is a nonzero invariant of degree M and obtain $\delta(n_1, \dots, n_d) \leqslant M$. $\square$

Corollary 2.4.2. *Let $d \geqslant 3$ be odd. Then*

$$\delta_d(n) \geqslant n\lceil n^{1/(d-1)} \rceil.$$

Corollary 2.4.3. *The inequality $\delta_d(n) \leqslant \delta_{d-2}(n)$ holds for $d > 3$.*

Proof. Note that $\delta_2(n) = n$ (whose invariant is the ordinary determinant). Then the upper bound gives:

$$\delta_d(n) \leqslant \operatorname{lcm}(\delta_2(n), \delta_{d-2}(n)) = \operatorname{lcm}(n, \delta_{d-2}(n)) = \delta_{d-2}(n).$$

as needed. $\square$

In particular, $\delta_d(n) \leqslant \delta_3(n)$ for all odd $d \geqslant 3$. Now is shown an upper bound for $\delta_3(n)$.

Lemma 2.4.4 (cf. [41, Thm. 5.9])**.** $\delta_3(n) \leqslant n^2$.

Proof. Recall that $\dim \operatorname{Inv}_3(n)_{n^2} = g(n\times n, n\times n, n\times n) > 0$. Hence there exists an invariant of degree n^2 and so $\delta_3(n) \leqslant n^2$. $\square$

An explicit construction of an invariant of degree n^2 in § a) is discussed; it is conditional on the Alon–Tarsi conjecture for even n.

Note that Theorem 2.3.2 (i) now follows from Corollary 2.4.2, Corollary 2.4.3, and Lemma 2.4.4.

2.4.2 Dimension sequences

Recall

$$g_d(n,k) := \dim \operatorname{Inv}_d(n)_{kn} = g(\underbrace{n \times k, \dots, n \times k}_{d \text{ times}}).$$

Theorem 2.4.5. *Let $d \geqslant 3$ be odd. The following properties hold.*

(i) Bounds:

$$g_d(n,k) \leqslant \binom{k^d}{nk}.$$

In particular, $g_d(n,k) = 0$ for $n > k^{d-1}$.

(ii) Symmetry:

$$g_d(n,k) = g_d(k^{d-1} - n, k) \text{ for } n \in [0, k^{d-1}].$$

In particular, $g_d(0,k) = g_d(k^{d-1},k) = 1$.

(iii) Positivity: For fixed k, if the sequence $\{g_d(n,k)\}_{n=0}^{k^{d-1}}$ is positive for $d = 3$, then it is positive for all odd $d > 3$.

Lemma 2.4.6. *Let $d \geqslant 5$ be odd. The following formula holds:*

$$g_d(n,k) = \sum_{\mu^{(2)},\dots\mu^{(d-2)}} g(\mu^{(1)}, k\times n, \mu^{(2)}) \\ g(\mu^{(2)}, k\times n, \mu^{(3)}) \cdots g(\mu^{(d-2)}, k\times n, \mu^{(d-1)}), \tag{2.9}$$

where $\mu^{(1)} = k \times n$ and $\mu^{(d-1)} = n \times k$.

Proof. By conjugation and self duality obtain $[n\times k]\otimes[n\times k] \simeq [k\times n]\otimes[k\times n]$ and hence obtain that

$$\begin{aligned} g_d(n,k) &= \text{mult. } [n\times k] \text{ in } \underbrace{[n\times k]\otimes\cdots\otimes[n\times k]}_{(d-1)\text{ times}} \\ &= \text{mult. } [n\times k] \text{ in } \underbrace{[k\times n]\otimes\cdots\otimes[k\times n]}_{(d-1)\text{ times}}. \end{aligned}$$

The formula (2.9) then follows by iteratively decomposing the last tensor product expression. □

For a partition λ denote the *width* $w(\lambda) := \ell(\lambda') = \lambda_1$.

Lemma 2.4.7 (Rectangular bound)**.** *Let $\mu^{(1)} = k\times n, \mu^{(2)}, \dots, \mu^{(d-2)}, \mu^{(d-1)} = n \times k$ be partitions such that*

$$g(\mu^{(i)}, k\times n, \mu^{(i+1)}) > 0 \text{ for } i = 1,\dots,d-2.$$

Then $\mu^{(i)} \subseteq k^i \times k^{d-i}$ for $i = 1,\dots,d-1$.

Proof. First, prove that $\ell(\mu^{(i)}) \leqslant k^i$ by induction on $i = 1,\dots,d-2$. For $i = 1$ it is known that $\ell(k\times n) = k^1$. Now assume $\ell(\mu^{(i)}) \leqslant k^i$. By Lemma 2.1.1(f) the positivity $g(\mu^{(i)}, k\times n, \mu^{(i+1)}) > 0$ implies that

$$\ell(\mu^{(i+1)}) \leqslant \ell(\mu^{(i)}) \cdot \ell(k\times n) \leqslant k^{i+1}$$

as desired.

Let us now show that $w(\mu^{(i)}) \leqslant k^{d-i}$ by reverse induction on $i = d-1,\dots,1$. For $i = d-1$ it is known that $w(n\times k) = k$. Now assume $w(\mu^{(i)}) \leqslant k^{d-i}$. Again by Lemma 2.1.1(f) the positivity $g(\mu^{(i-1)}, k\times n, \mu^{(i)}) > 0$ implies

$$w(\mu^{(i-1)}) \leqslant |\mu^{(i)} \cup (k\times n)| \leqslant |(k^i \times k^{d-i}) \cup (k\times n)| \leqslant k^{d-i+1}$$

and the proof follows. □

of Theorem 2.4.5. (i) The Kronecker coefficient $g_d(n,k)$ is bounded from above by the number of $d \times nk$ balanced tables with entries from $[k]$ and distinct columns, where the order or columns is not important. The maximal number of distinct columns with entries from $[k]$ is at most k^d and hence the number of such $d \times nk$ balanced tables (as an ordered set of columns) is bounded above by $\binom{k^d}{nk}$.

Remark 2.4.8. Let us also note that this is a quite rough bound; using balanced tables it is also easy to show that

$$g_d(n,k) \leqslant \frac{1}{k!^{d-1}} \binom{nk}{n,\ldots,n}^{d-1} = \left(\frac{(nk)!}{n!^k k!}\right)^{d-1}.$$

(ii) Let $\mu^{(1)} = k \times n, \mu^{(2)}, \ldots, \mu^{(d-2)}, \mu^{(d-1)} = n \times k$ be partitions representing a nonzero term in the expansion (2.9) of $g_d(n,k)$. By Lemma 2.4.7 holds $\mu^{(i)} \subseteq k^i \times k^{d-i}$. Define a mapping of every such $(d-1)$-tuple of partitions $\mu^{(i)}$ to a $(d-1)$-tuple of partitions $\nu^{(i)}$ from the similar expansion of $g_d(k^{d-1}-n,k)$.

Let $\nu^{(i)} = k^i \times k^{d-i} -_{k^i} \mu^{(i)}$ for $i = 1, \ldots, d-1$. In particular, $\nu^{(1)} = k \times (k^{d-1} - n)$ and $\nu^{(d-1)} = (k^{d-1}-n) \times k$. In Lemma 2.1.1(e) set $(a,b,c) \to (k^i, k, k^{d-i-1})$ and apply conjugation to obtain

$$\begin{aligned} & g(\mu^{(i)}, k \times n, \mu^{(i+1)}) = g((\mu^{(i)})', n \times k, \mu^{(i+1)}) \\ & = g(k^{d-i} \times k^i -_{k^{d-i}} (\mu^{(i)})', k^{d-1} \times k -_{k^{d-1}} n \times k, k^{i+1} \times k^{d-i-1} -_{k^{i+1}} \mu^{(i+1)}) \\ & = g(k^i \times k^{d-i} -_{k^i} \mu^{(i)}, k \times (k^{d-1}-n), k^{i+1} \times k^{d-i-1} -_{k^{i+1}} \mu^{(i+1)}) \\ & = g(\nu^{(i)}, k \times (k^{d-1}-n), \nu^{(i+1)}) \end{aligned}$$

for each $i = 1, \ldots, d-2$. Therefore, nonzero terms in the expansions (2.9) for $g_d(n,k)$ and $g_d(k^{d-1}-n,k)$ can be bijectively mapped to each other, which implies the symmetry $g_d(n,k) = g_d(k^{d-1}-n,k)$ for all $n \in [0, k^{d-1}]$.

(iii) Proceed by induction on odd $d \geqslant 3$. The base case $d = 3$ is the given condition. Assume $d \geqslant 3$ is odd and $g_d(n,k) > 0$ for all $n \in [0, k^{d-1}]$.

Let us show that $g_{d+2}(n,k) > 0$ for all $n \in [0, k^{d+1}]$. Fix $n \in [0, k^{d-1}]$ and expand $g_d(n,k)$ as in (2.9). Since $g_d(n,k) > 0$ there are partitions $\mu^{(1)} = k \times n, \mu^{(2)}, \ldots, \mu^{(d-2)}, \mu^{(d-1)} = n \times k$ such that

$$g(\mu^{(1)}, k \times n, \mu^{(2)})\, g(\mu^{(2)}, k \times n, \mu^{(3)}) \cdots g(\mu^{(d-2)}, k \times n, \mu^{(d-1)}) > 0. \tag{2.10}$$

For each $\ell = 0, \ldots, k^2-1$ and $N = \ell k^{d-1} + n$ prove the positivity of $g_{d+2}(N,k)$. Note that N ranges in $[\ell k^{d-1}, (\ell+1)k^{d-1}]$ as n ranges in $[0, k^{d-1}]$. Set $\nu^{(i)} = k^i \times \ell k^{d-i} + \mu^{(i)}$ for $i = 1, \ldots, d-1$. By Lemma 2.4.7 holds $\mu^{(i)} \subseteq k^i \times k^{d-i}$. Applying Lemma 2.1.1(e) for ℓ times with the parameters $(a,b,c) = (k^i, k, k^{d-i-1})$ obtain

$$\begin{aligned} g(\mu^{(i)}, k \times n, \mu^{(i+1)}) = g(&\mu^{(i)} + k^i \times \ell k^{d-i}, \\ & k \times n + k \times \ell k^{d-1}, \end{aligned}$$

$$\mu^{(i+1)} + k^{i+1} \times \ell k^{d-i-1})$$
$$= g(\nu^{(i)}, k \times N, \nu^{(i+1)}) > 0$$

for all $i = 1, \ldots, d-2$ and $\nu^{(i)} \subseteq k^i \times k^{d+2-i}$. Denote $m = k^{d-1} - n \in [0, k^{d-1}]$. Then

$$\nu^{(d-1)} = k^{d-1} \times \ell k + n \times k, \qquad (\nu^{(d-1)})' = (\ell+1)k \times n + \ell k \times m.$$

Let us now define the remaining two partitions for $i = d, d+1$:

$$\nu^{(d)} = (\ell \times mk + (\ell+1) \times nk)',$$
$$\nu^{(d+1)} = N \times k, \qquad (\nu^{(d+1)})' = k \times (\ell+1)n + k \times \ell m.$$

Then the claim is that the following two coefficients are also positive:

$$g(\nu^{(d-1)}, k \times N, \nu^{(d)}) > 0, \quad g(\nu^{(d)}, k \times N, \nu^{(d+1)}) > 0,$$

or by conjugating the arguments in both coefficients one can rewrite them as

$$\begin{aligned} g(&(\ell+1)k \times n + \ell k \times m, \\ &k \times (\ell+1)n + k \times \ell m, \\ &(\ell+1) \times kn + \ell \times km) > 0, \end{aligned} \tag{2.11}$$
$$\begin{aligned} g(&(\ell+1) \times kn + \ell \times km, \\ &k \times (\ell+1)n + k \times \ell m, \\ &k \times (\ell+1)n + k \times \ell m) > 0. \end{aligned} \tag{2.12}$$

Indeed, in Lemma 2.1.1(e) set $(a,b,c) \to (n,k,\ell+1)$ and $(a,b,c) \to (m,k,\ell)$ and apply the semi-group property to the resulting coefficients to obtain the positivity (2.11). Further, since $\ell \in [0, k^2-1]$, by assumption the coefficients $g((\ell+1) \times k, k \times (\ell+1), k \times (\ell+1))$ and $g(\ell \times k, k \times \ell, k \times \ell)$ are positive, and hence $g(\ell \times km, k \times \ell m, k \times \ell m)$ and $g((\ell+1) \times kn, k \times (\ell+1)n, k \times (\ell+1)n)$ are positive as well by applying the semi-group property to itself m and n times, respectively. Combining the last two coefficients and the semi-group property the positivity of (2.12) is obtained. So, the statement $g(\nu^{(i)}, k \times N, \nu^{(i+1)})$ is positive for all $i \in [1, d+1]$ is shown, hence the expansion (2.9) of $g_{d+2}(N,k)$ contains a positive term. This completes the induction step. □

A direct computation (see Appendix B) shows that $\{g_3(n,k)\}_{n=0}^{k^2}$ is positive for $k = 2, 4$. Therefore, the following result is obtained.

Corollary 2.4.9. *For all odd $d \geqslant 3$ and $k = 2, 4$, the sequence $\{g_d(n,k)\}_{n=0}^{k^{d-1}}$ is positive.*

It is conjectured that the sequence $\{g_d(n,k)\}_{n=0}^{k^{d-1}}$ is positive for (a) odd $d \geqslant 5$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$g_3(n,4)$	1	1	1	2	5	6	13	14	18	14	13	6	5	2	1	1	1
$g_5(n,2)$	1	1	5	11	35	52	112	130	166	130	112	52	35	11	5	1	1

Figure 2.4 – Examples of degree sequences displaying unimodality.

and all k, and (b) $d = 3$ and even k. Note that for odd k one have $g_3(2,k) = g_3(k^2-2,k) = 0$ (e.g. [50]) which seem to be the only exceptional cases; in particular, this gives $\delta_3(n) \geqslant n(k+1)$ for odd $n = k^2-2$. Furthermore, it seems reasonable to conjecture that for (a), (b) the sequence $\{g_d(n,k)\}_{n=0}^{k^{d-1}}$ (besides being symmetric and positive) is also unimodal. See Fig. 2.4, for example. Some more related computational data is provided in Appendix B.

Lemma 2.4.10. *Let $d \geqslant 3$ be odd. Assume for fixed k the sequence $\{g_d(n,k)\}_{n=0}^{k^{d-1}}$ is positive. Then*
(i) $\delta_d(n) = nk$ for all $(k-1)^{d-1} < n \leqslant k^{d-1}$, and
(ii) $\delta_d(n) \in \{n(k-1), nk\}$ for all $(k-2)^{d-1} < n \leqslant (k-1)^{d-1}$.

Proof. Since $\dim \mathrm{Inv}_d(n)_{nk} = g_d(n,k) > 0$ there is an invariant of degree nk for all $n \leqslant k^{d-1}$. Hence, $\delta_d(n) \leqslant nk$. (i) On the other hand, by Corollary 2.4.2 have $\delta_d(n) \geqslant nk$ for $(k-1)^{d-1} < n$. Hence, $\delta_d(n) = nk$ for $(k-1)^{d-1} < n \leqslant k^{d-1}$. (ii) By Corollary 2.4.2 have $\delta_d(n) \geqslant n(k-1)$ for $(k-2)^{d-1} < n$. □

Corollary 2.4.11. *Let $d \geqslant 3$ be odd. Then*

$$\delta_d(n) = n\lceil n^{1/(d-1)}\rceil \text{ for } n \in \{1,\ldots,2^{d-1}\} \cup \{3^{d-1},\ldots,4^{d-1}\} \cup \{k^{d-1}-1, k^{d-1} : k \in \mathbb{N}_{\geqslant 2}\},$$

and also $\delta_d(n) \in \{3n, 4n\}$ for $n \in \{2^{d-1}+1,\ldots,3^{d-1}\}$.

Proof. Note that $\delta_d(k^{d-1}) \geqslant k^d$ and $\delta_d(k^{d-1}-1) \geqslant k^d - k$ by Corollary 2.4.2. On the other hand, $\dim \mathrm{Inv}_d(k^{d-1})_{k^d} = g_d(k^{d-1},k) = 1$ and $\dim \mathrm{Inv}_d(k^{d-1}-1)_{k^d-k} = g_d(k^{d-1}-1,k) = g_d(1,k) = 1$ (the last fact can be easily seen using (2.9) and trivial characters) and so there are invariants of degrees k^d and k^d-k. Hence $\delta_d(k^{d-1}) = k^d$ and $\delta_d(k^{d-1}-1) = k^d - k$. The rest follows from Corollary 2.4.9 and Lemma 2.4.10. □

This establishes part of Theorem 2.3.2 (ii).

Corollary 2.4.12. *Let $n > 1$ be fixed. The sequence $\delta_3(n) \geqslant \delta_5(n) \geqslant \cdots \geqslant \delta_d(n) \geqslant \cdots$ stabilizes to $2n$ for $d \geqslant 1 + \log_2 n$.*

Proof. Monotonicity of the sequence is established in Corollary 2.4.3. The stabilization limit follows from the above fact that

$$\delta_d(n) = 2n$$

for $1 < n \leqslant 2^{d-1}$. □

2.4.3 Conclusion

This section has analyzed the degree sequences and dimensions of invariant polynomials, focusing on determining bounds for the smallest degrees of these invariants. The notation $\delta(n_1, \dots, n_d)$ was introduced to denote the smallest degrees of invariants, with $\delta_d(n) = \delta(n, \dots, n)$ (d times).

Key findings include:

1 **Lower and Upper Bounds**: For odd $d \geqslant 3$, the lower bound is:

$$\delta(n_1, \dots, n_d) \geqslant \lceil (n_1 \dots n_d)^{1/(d-1)} \rceil,$$

and a recursive upper bound is:

$$\delta(n_1, \dots, n_d) \leqslant \mathrm{lcm}(\delta(n_1, \dots, n_\ell), \delta(n_{\ell+1}, \dots, n_d)).$$

For $d \geqslant 3$, the upper bound is:

$$\delta_d(n) \leqslant n^2.$$

2 **Dimension Sequences**: The dimension sequences of invariant polynomials, denoted by $g_d(n, k)$, exhibit important properties such as bounds, symmetry, and positivity. For example:

$$g_d(n, k) \leqslant \binom{k^d}{nk}.$$

Symmetry and positivity results show that if the sequence $\{g_d(n, k)\}_{n=0}^{k^{d-1}}$ is positive for $d = 3$, then it remains positive for all odd $d > 3$.

3 **Degree Sequence Monotonicity**: The sequence $\delta_3(n) \geqslant \delta_5(n) \geqslant \cdots \geqslant \delta_d(n) \geqslant \cdots$ stabilizes to $2n$ for $d \geqslant 1 + \log_2 n$. This indicates that for sufficiently large d, the degree of the smallest invariant polynomial reaches a stable value.

4 **Specific Cases and Conjectures**: Specific results for certain cases include:

$$\delta_d(n) = n\lceil n^{1/(d-1)} \rceil \text{ for } n \in \{1, \dots, 2^{d-1}\} \cup \{3^{d-1}, \dots, 4^{d-1}\} \\ \cup \{k^{d-1} - 1, k^{d-1} : k \in \mathbb{N}_{\geqslant 2}\},$$

and

$$\delta_d(n) \in \{3n, 4n\} \text{ for } n \in \{2^{d-1} + 1, \ldots, 3^{d-1}\}.$$

These results provide a deeper understanding of the structure and properties of invariant polynomials, offering valuable insights for future research in this area. The established bounds and properties lay the groundwork for further explorations into the combinatorial and algebraic aspects of invariant theory.

2.5 Fundamental invariants and Latin hypercubes

2.5.1 Explicit fundamental invariants

For $n = k^{d-1}$ consider the $d \times k^d$ balanced table $T = (T_{ij})$ given by

$$T_{ij} = \left\lfloor \frac{j-1}{k^i} \right\rfloor \pmod{k} + 1$$

i.e. the columns of T are the elements of $[k]^d$ listed in lexicographical order. Also for $n = k^{d-1} - 1$ let $\widetilde{T}$ be the $d \times (k^d - k)$ balanced table obtained by removing from T the columns $(1, \ldots, 1)^T, \ldots, (k, \ldots, k)^T$.

For example, for $d = 3$ and $k = 2$ one have

$$T = \begin{pmatrix} 11112222 \\ 11221122 \\ 12121212 \end{pmatrix}, \qquad \widetilde{T} = \begin{pmatrix} 111222 \\ 122112 \\ 212121 \end{pmatrix}.$$

The table T has the property that in every row each element in $[k]$ appears k^{d-1} times and its k^d columns are all distinct (out of k^d possible columns), which is a unique balanced table (up to a permutation of columns) with such a property. In addition, the columns of T are ordered lexicographically.

Let us denote

$$F_{d,k} := \Delta_T, \qquad \widetilde{F}_{d,k} := \Delta_{\widetilde{T}}.$$

the corresponding invariants.

Theorem 2.5.1. *Let $d \geqslant 3$ be odd. Then*

(i) $F_{d,k} \in \mathrm{Inv}_d(k^{d-1})_{k^d}$ is the unique (up scale) fundamental invariant of degree $\delta_d(k^{d-1}) = k^d$.

(ii) $\widetilde{F}_{d,k} \in \mathrm{Inv}_d(k^{d-1} - 1)_{k^d - k}$ is the unique (up scale) fundamental invariant of degree $\delta_d(k^{d-1} - 1) = k^d - k$.

Proof. The existence and uniquencess of such fundamental invariants follows from the fact that $\dim \mathrm{Inv}_d(k^{d-1})_{k^d} = g_d(k^{d-1}, k) = 1$ and $\dim \mathrm{Inv}_d(k^{d-1} - 1)_{k^d - k} = g_d(k^{d-1} - 1, k) = g_d(1, k) = 1$ shown in the previous section. (i) This invariant written as Δ_T corresponds to a $d \times k^d$ balanced table with distinct columns. Note that there is a unique such table when one disregards the order of columns due to Lemma 2.3.15 and Corollary 2.3.16. (ii) A nonzero Δ polynomial from $\mathrm{Inv}_d(k^{d-1} - 1)_{k^d - k}$ must be indexed by a $d \times (k^d - k)$ table obtained from the full table T by removing k columns which leave the resulting table balanced, i.e. in these columns every entry from $[k]$ appears once in every row. Note that if in any row of T have the values of i and j from $[k]$ switched, the resulting Δ polynomial will not change (by definition, as it corresponds to

Δ_S with the same tuple of set partitions S). Hence by performing these swap operations it may be assumed that the k columns removed from T are exactly $(1,\dots,1)^T,\dots,(k,\dots,k)^T$, and the resulting nonzero polynomial is $\Delta_{\tilde{T}}$. □

Remark 2.5.2. The fundamental invariant $F_{3,k}$ was introduced in [41], and $\tilde{F}_{3,k}$ (formulated via obstruction designs) was introduced in [51].

2.5.2 Latin hypercubes

a) An invariant of degree k^2

A *Latin square* of length k is an $k \times k$ matrix whose every row and column is a permutation of $[k]$. Let $\mathcal{C}_2(k)$ be the set of Latin squares of length k. The *sign* of a Latin square L is the product of signs of permutations in rows and columns, denoted by $\operatorname{sgn}(L)$. The *Alon–Tarsi number* $AT(k)$ is the following signed sum over Latin squares:

$$AT(k) := \sum_{L \in \mathcal{C}_2(k)} \operatorname{sgn}(L).$$

It is not difficult to show that $AT(k) = 0$ for odd k. The Alon–Tarsi conjecture [43] states that $AT(k) \neq 0$ for even k.

There is also an equivalent conjecture due to Huang and Rota [52], which states the sum of *column signs* (i.e. the product of signs of columns) of Latin squares in $\mathcal{C}_2(k)$ is also nonzero for even k.

Denote by $I_k := (\delta_{i_1 i_2} \cdots \delta_{i_1 i_d})_{i_1,\dots,i_d \in [k]}$ the unit tensor.

Proposition 2.5.3. *Let $d \geqslant 3$ be odd, k be even and T be $d \times k^2$ balanced table given by*

$$T = \begin{pmatrix} 1^k\, 2^k \cdots k^k \\ \cdots \\ 1^k\, 2^k \cdots k^k \\ e_k\, e_k \cdots e_k \end{pmatrix}$$

where denotes $j^k = \underbrace{jj\dots j}_{k \text{ times}}$ and $e_k = 123\dots k$. Then $\Delta_T(I_k) \neq 0$ is equivalent to the Alon-Tarsi conjecture $AT(k) \neq 0$. In particular, if $AT(k) \neq 0$ then $\Delta_T \in \operatorname{Inv}_d(k)_{k^2}$ is a nonzero invariant of degree k^2.

Proof. Consider the evaluation of Δ_T at identity tensor. Each nonzero term in the sum (2.6) corresponds to a map $\sigma : [k^2] \to [k]$ such that $(\sigma(1 + (i-1)k), \dots, \sigma(ik)) \in S_k$ for each $i \in [k]$ (from the first $d-1$ rows of T), and also $(\sigma(i), \sigma(i+k), \dots, \sigma(i + (k-1)k)) \in S_k$ for each $i \in [k]$ (from the last row of T). Then let $C = (C_{ij})$ be given by $C_{ij} = \sigma(i + (j-1)k)$. The resulting matrix is indeed a Latin square, since each column and row is a permutation. Let $r_1, \ldots, r_k$ and $c_1, \ldots, c_k$ be rows and columns of C, respectively; each of them is a permutation of S_k. Then

$$\operatorname{sgn}(\sigma) = \operatorname{sgn}(r_1 \cdots r_k)^{d-1} \operatorname{sgn}(c_1 \cdots c_k) = \operatorname{sgn}(c_1 \cdots c_k)$$

and so the sign corresponding to σ in the sum is exactly the column sign of C, i.e. the product of signs of columns. Hence, $\Delta_T(I_k)$ is the sum of these signs and one have $\Delta_T(I_k) \neq 0$ iff $AT(k) \neq 0$ (which is the equivalence between the Huang–Rota and Alon–Tarsi conjectures). □

Remark 2.5.4. The Alon–Tarsi conjecture $AT(k) \neq 0$ is known to hold for $k = p \pm 1$ where $p \geqslant 3$ is a prime [53, 54].

b) Magic sets, Latin cubes and signs

Elements of the box $[k]^d$ are refered to as *cells*. A *slice* of $[k]^d$ is a subset of all cells with fixed ℓ-th coordinate (called *direction*) for some $\ell \in [d]$. A *diagonal* of $[k]^d$ is a subset of size k with no two cells lying in the same slice.

A *magic set* is a subset of $[k]^d$ which has an equal number of elements in every slice of $[k]^d$. It is possible to represent a magic set T as a *magic hypermatrix* with 1 at cells corresponding to elements of T and 0 elsewhere, which is a natural generalization of *magic squares*.

A *d-dimensional partial Latin hypercube* of length k and cardinality M is a function $C : [k]^d \to \{0, 1, \ldots, n = M/k\}$ satisfying the following two conditions:

- the cells $\{a \in [k]^d : C(a) \neq 0\}$ form a magic set of cardinality M, and
- for each slice A of $[k]^d$, the nonzero values $C(a) \neq 0$ written in lexicographical order of $a \in A$, form a permutation of $[n]$ which is called *C-permutation* of this slice.

The underlying magic set of C is called its *type*. The set of d-dimensional partial Latin hypercubes of type T and length k is denoted by $\mathcal{C}_d(k, T)$. If $T = [k]^d$, then $M = nk = k^d$ and the elements of $\mathcal{C}_d(k, T)$ are called (full) Latin hypercubes. The set of Latin hypercubes of length k is denoted by $\mathcal{C}_d(k) = \mathcal{C}_d(k, [k]^d)$. See Fig. 2.5 with some examples.

Definition 2.5.5. Let $C \in \mathcal{C}_d(k, T)$ be a partial Latin hypercube of length k and type $T \subseteq [k^d]$ with $|T| = nk$. Define the following sign functions:

- *the directional sign* $\operatorname{sgn}_\ell(C)$ is the product of signs of C-permutations of the slices in the direction $\ell \in [d]$;
- *the full sign* $\operatorname{sgn}(C) := \operatorname{sgn}_1(C) \cdots \operatorname{sgn}_d(C)$;
- *the symbol sign* $\operatorname{ssgn}(C)$ is the product of *subsigns* of each $i \in [n]$ defined as follows. Let $\operatorname{diag}_i(C) := \{a \in [k]^d : C(a) = i\}$. Since $i \in [n]$ is present in each slice exactly once, then $\operatorname{diag}_i(C)$ is a diagonal and hence can be described by

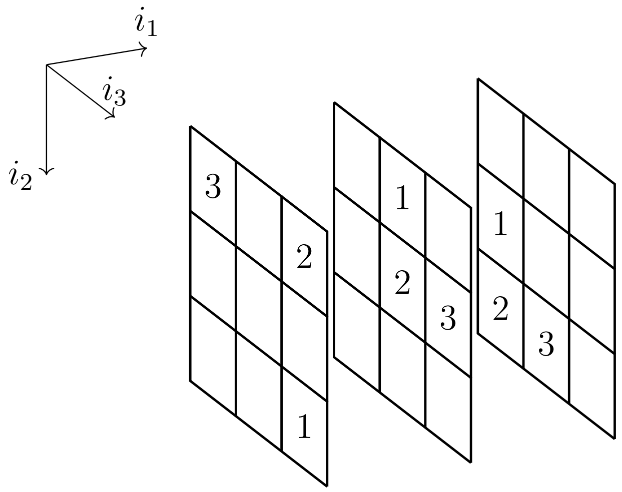

Figure 2.5 – A partial Latin cube $C(i_1, i_2, i_3)$ of length $k = 3$ and cardinality $M = 9$; its type (cells of nonzero values) is a magic set.

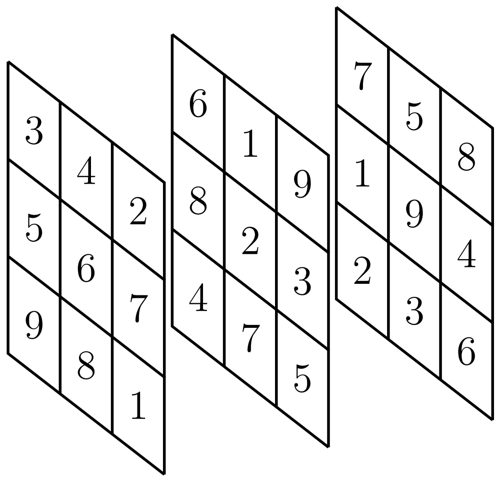

Figure 2.6 – A Latin cube in $\mathcal{C}_3(3)$.

permutations $\pi_2, \ldots, \pi_d$ of $[k]$ so that $\text{diag}_i(C) = \{(j, \pi_2(j), \ldots, \pi_d(j)) : j \in [k]\}$. Then the subsign of i is $\text{sgn}(\pi_2) \cdots \text{sgn}(\pi_d)$.

Example 2.5.6. Let us compute the signs of the partial Latin cube $C \in \mathcal{C}_3(3, T)$ shown on Fig. 2.5 which is given by

$$C(1,3,3) = C(2,1,2) = C(3,2,1) = 1,$$
$$C(1,1,3) = C(2,2,2) = C(3,3,1) = 2,$$
$$C(1,1,1) = C(2,2,3) = C(3,3,2) = 3,$$

and $C(i_1, i_2, i_3) = 0$ otherwise. Its slices in: direction 1 have C-permutations $(3,2,1)$, $(1,2,3)$, $(1,2,3)$ and $\text{sgn}_1(C) = -1$; direction 2 have C-permutations $(3,2,1)$, $(2,3,1)$, $(1,2,3)$ and $\text{sgn}_2(C) = -1$; direction 3 have C-permutations $(3,1,2)$, $(1,2,3)$, $(2,1,3)$ and $\text{sgn}_3(C) = 1$; and hence, $\text{sgn}(C) = 1$.

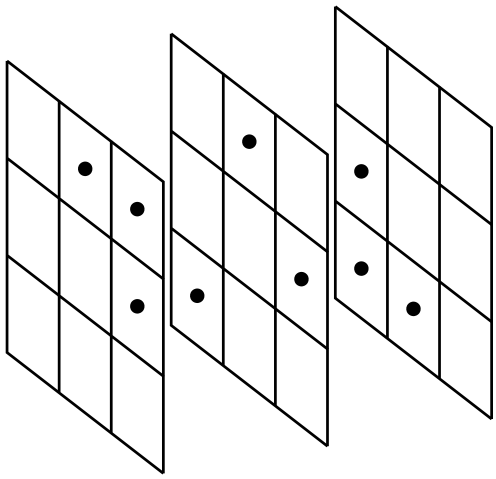

Figure 2.7 – A magic set which does not give rise to a partial Latin cube.

To compute the symbol sign, one have: for symbol $1, 2, 3$ the diagonals are:
$\text{diag}_1(C) = \{(1,3,3), (2,1,2), (3,2,1)\}$ is of sign $\text{sgn}(3,1,2)\,\text{sgn}(3,2,1) = -1$;
$\text{diag}_2(C) = \{(1,1,3), (2,2,2), (3,3,1)\}$ is of sign $\text{sgn}(1,2,3)\,\text{sgn}(3,2,1) = -1$;
$\text{diag}_3(C) = \{(1,1,1), (2,2,3), (3,3,2)\}$ is of sign $\text{sgn}(1,2,3)\,\text{sgn}(1,3,2) = -1$;
and hence, $\text{ssgn}(C) = -1$.

Example 2.5.7. Let us compute the signs of the Latin cube $C \in \mathcal{C}_3(3)$ shown in Fig. 2.6 which is presented by slices in direction 1 (as matrices) given by

$$C(1,\cdot,\cdot) = \begin{pmatrix} 3 & 4 & 2 \\ 5 & 6 & 7 \\ 9 & 8 & 1 \end{pmatrix}, \quad C(2,\cdot,\cdot) = \begin{pmatrix} 6 & 1 & 9 \\ 8 & 2 & 3 \\ 4 & 7 & 5 \end{pmatrix}, \quad C(3,\cdot,\cdot) = \begin{pmatrix} 7 & 5 & 8 \\ 1 & 9 & 4 \\ 2 & 3 & 6 \end{pmatrix}.$$

Then

$$\text{sgn}_1(C) = \text{sgn}(3,4,2,5,6,7,9,8,1)\,\text{sgn}(6,1,9,8,2,3,4,7,5)\,\text{sgn}(7,5,8,1,9,4,2,3,6)$$
$$\text{sgn}_2(C) = \text{sgn}(3,4,2,6,1,9,7,5,8)\,\text{sgn}(5,6,7,8,2,3,1,9,4)\,\text{sgn}(9,8,1,4,7,5,2,3,6)$$
$$\text{sgn}_3(C) = \text{sgn}(3,5,9,6,8,4,7,1,2)\,\text{sgn}(4,6,8,1,2,7,5,9,3)\,\text{sgn}(2,7,1,9,3,5,8,4,6)$$

and hence, $\text{sgn}(C) = 1$. To compute the symbol sign one have:
$\text{diag}_1(C) = \{(1,3,3), (2,1,2), (3,2,1)\}$, subsign of 1 is $\text{sgn}(312)\,\text{sgn}(321) = -1$,
$\text{diag}_2(C) = \{(1,1,3), (2,2,2), (3,3,1)\}$, subsign of 2 is $\text{sgn}(123)\,\text{sgn}(321) = -1$,
$\text{diag}_3(C) = \{(1,1,1), (2,2,3), (3,3,2)\}$, subsign of 3 is $\text{sgn}(123)\,\text{sgn}(132) = -1$,
$\text{diag}_4(C) = \{(1,1,2), (2,3,1), (3,2,3)\}$, subsign of 4 is $\text{sgn}(132)\,\text{sgn}(213) = 1$,
$\text{diag}_5(C) = \{(1,2,1), (2,3,3), (3,1,2)\}$, subsign of 5 is $\text{sgn}(231)\,\text{sgn}(132) = 1$,
$\text{diag}_6(C) = \{(1,2,2), (2,1,1), (3,3,3)\}$, subsign of 6 is $\text{sgn}(213)\,\text{sgn}(213) = 1$,
$\text{diag}_7(C) = \{(1,2,3), (2,3,2), (3,1,1)\}$, subsign of 7 is $\text{sgn}(231)\,\text{sgn}(321) = 1$,
$\text{diag}_8(C) = \{(1,3,2), (2,2,1), (3,1,3)\}$, subsign of 8 is $\text{sgn}(321)\,\text{sgn}(213) = 1$,

$\mathrm{diag}_9(C) = \{(1,3,1),(2,1,3),(3,2,2)\}$, subsign of 9 is $\mathrm{sgn}(312)\,\mathrm{sgn}(132) = -1$, and hence $\mathrm{ssgn}(C) = 1$. Note that $\mathrm{sgn}_1(C)^2\,\mathrm{sgn}_2(C)\,\mathrm{sgn}_3(C)\,\mathrm{ssgn}(C) = -1$ which is as in Corollary 2.5.19 below.

Example 2.5.8. It can also be noted that not every magic set gives rise to a Latin hypercube. For example, the set

$$T = \left\{ \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}1\\1\\3\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\1\\2\end{pmatrix}, \begin{pmatrix}2\\2\\3\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}3\\3\\1\end{pmatrix}, \begin{pmatrix}3\\3\\2\end{pmatrix} \right\}$$

(where elements represented as columns) shown in Fig. 2.7 is a magic set of length $k = 3$ and cardinality $M = 9$ but it cannot be decomposed as a disjoint union of diagonals, and hence there exist no partial Latin cubes of type T. Note that for $d = 2$ every magic square can be decomposed as a union of diagonals (transversals), which is (a special case of) Birkhoff's theorem.

c) d-dimensional Alon–Tarsi numbers

Let us denote

$$AT_d(k,T) := \sum_{C \in \mathcal{C}_d(k,T)} \mathrm{sgn}(C), \qquad AT_d(k) := AT_d(k,[k]^d).$$

Note that $AT_2(k) = AT(k)$ is the Alon–Tarsi number.

Proposition 2.5.9. *Let $\Delta_T \in \mathrm{Inv}_d(n)_{nk}$, where T is a $d \times nk$ balanced table with distinct columns ordered lexicographically. Then*

$$\Delta_T(I_n) = AT_d(k,T), \tag{2.13}$$

where in r.h.s. T is viewed as a set of its columns.

Proof. Let us show a bijection between nonzero terms of $\Delta_T(I_n)$ and partial Latin hypercubes. Let $\sigma : [nk] \to [n]$ be a map corresponding to a nonzero term in the expansion (2.6) of $\Delta_T(I_n)$ (note that in this expansion must have $\sigma_1 = \cdots = \sigma_d = \sigma$ as evaluated at the unit tensor). Let $T_j \in [k]^d$ be the j-th column of T for $j \in [nk]$. Construct a partial Latin hypercube C_σ by setting $C_\sigma(T_j) = \sigma(j)$. (At all remaining cells C is 0.) One can see that the resulting function $C_\sigma : [k]^d \to \{0,\ldots,n\}$ is indeed a partial Latin hypercube of length k and type T, i.e. $C_\sigma \in \mathcal{C}_d(k,T)$. Since the columns of T are ordered lexicographically, one also has $\mathrm{sgn}_T(\sigma) = \mathrm{sgn}(C_\sigma)$. Conversely, given $C \in \mathcal{C}_d(k,T)$ set $\sigma(j) = C(T_j)$. This gives a sign preserving bijection. □

Corollary 2.5.10. *For the fundamental invariant $F_{d,k}$ following holds*

$$F_{d,k}(I_{k^{d-1}}) = \sum_{C \in \mathcal{C}_d(k)} \operatorname{sgn}(C) = AT_d(k). \tag{2.14}$$

The following statement is a generalization of Proposition 2.5.3.

Proposition 2.5.11. *Let $d \geqslant 3$ odd, k be even, and $\ell \leqslant d$ be odd, $n = k^{\ell-1}$. Let T be a $d \times k^\ell$ balanced table whose row $i \in [d]$ denoted as T_i is given by*

$$T_i = \begin{cases} (1^{k^{\ell-i}} \dots k^{k^{\ell-i}})^{k^{i-1}} & \text{if } i < \ell, \\ (e_k)^{k^{\ell-1}} & \text{otherwise,} \end{cases}$$

where $j^n = \underbrace{jj\dots j}_{n \text{ times}}$ and $e_k = 123\dots k$. Then $\Delta_T(I_n) = AT_\ell(k)$. In particular, if $AT_\ell(k) \neq 0$ then Δ_T is a nonzero invariant of degree k^ℓ.

Proof. Let $\sigma = (\sigma_1, \dots, \sigma_d)$ be a d-tuple of maps $[k^\ell] \to [k]$ corresponding to a nonzero term in the expansion (2.6) of $\Delta_T(I_n)$. Note that it must be $\sigma_\ell = \dots = \sigma_d$. Let H be a balanced table formed with the first ℓ rows of T and $\sigma' = (\sigma_1, \dots, \sigma_\ell)$, and H' be the table of the remaining $d - \ell$ rows. Note that H is a table of all possible $[k]^\ell$ columns and hence $\Delta_H = F_{\ell,k}$. Then

$$\operatorname{sgn}_T(\sigma) = \operatorname{sgn}_H(\sigma') \operatorname{sgn}_{H'}(\sigma_\ell)^{d-\ell} = \operatorname{sgn}_H(\sigma').$$

Hence there is a sign-preserving bijection between nonzero terms of the sum in the expansion of $\Delta_T(I_n)$ and nonzero terms in the expansion of $\Delta_H(I_n)$. By Corollary 2.5.10 one has $\Delta_T(I_n) = \Delta_H(I_n) = AT_\ell(k)$. □

Lemma 2.5.12. *Let C be a d-dimensional partial Latin hypercube of length k and cardinality nk. Let C' be a partial Latin hypercube obtained from C by exchanging its values $i \neq j$ from $[n]$. Then*

$$\operatorname{sgn}(C') = (-1)^{dk} \operatorname{sgn}(C).$$

Proof. The exchange results in a single transposition in each slice, and since there are dk slices, the proposition follows. □

Corollary 2.5.13. *Let d and k be odd and $n > 1$. For any $\Delta_T \in \operatorname{Inv}_d(n)_{kn}$ then $\Delta_T(I_n) = 0$. In particular, for any invariant $F \in \operatorname{Inv}_d(n)_{kn}$ holds $F(I_n) = 0$.*

Proof. By Lemma 2.5.12 the exchange of two values (say 1 and 2) defines the sign reversing involution on $\mathcal{C}_d(k, T)$, and hence all terms cancel out. □

Remark 2.5.14. On the other hand, it is known that there is an invariant Δ_T such that $\Delta_T(I_n) \neq 0$ since the unit tensor I_n is *semistable* for the action of G, cf. [44]. Hence its degree must be kn for some even $k \geqslant \delta_d(n)/n$.

In particular, $AT_d(k) = 0$ for odd k and d. Let us show that for $k = 2$ it is nonzero.

Proposition 2.5.15. *For all $d \geqslant 2$, holds $AT_d(2) > 0$.*

Proof. Let $C \in \mathcal{C}_d(2)$. Each diagonal of C has the form $\{a, \bar{a}\}$ for some $a \in [2]^d$, where $\bar{a} := (3, \ldots, 3) - a$. Hence for each $a \in [2]^d$ it must be $C(a) = C(\bar{a})$. Thus any two parallel slices contain permutations reverse to each other, namely, if π_1 is a permutation (collected lexicographically) in the slice 1 in the direction $i \in [d]$, then the permutation π_2 from the unique parallel slice 2 (also collected lexicographically) satisfies $\pi_2 = w_0 \circ \pi_1$, where w_0 is the reverse of the identity permutation of length 2^{d-1}. Hence,

$$\mathrm{sgn}_i(C) = \mathrm{sgn}(\pi_1)\,\mathrm{sgn}(\pi_2) = \mathrm{sgn}(w_0),$$

$$\mathrm{sgn}(C) = \mathrm{sgn}_1(C) \cdots \mathrm{sgn}_d(C) = \mathrm{sgn}(w_0)^d = (-1)^{2^{d-2}d} = 1$$

for all $d \geqslant 2$ and the result follows. □

Remark 2.5.16. In [41] it is also noted that computations show $AT_3(4) \neq 0$.

d) Relation between signs

For a sequence $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ let

$$\mathrm{inv}(a) := |\{(i, j) : a_i > a_j,\ 1 \leqslant i < j \leqslant n\}|, \qquad \mathrm{msgn}(a) := (-1)^{\mathrm{inv}(a)}.$$

be the number of inversions and the *multi-sign* of a. When $a \in S_n$ one have $\mathrm{msgn}(a) = \mathrm{sgn}(a)$.

Let $T = \{x_1, \ldots, x_{nk}\} \subseteq [k]^d$ be a magic set with the cells $x_1 < \ldots < x_{nk} \in [k]^d$ ordered lexicographically. Let $x_i = (x_i^{(1)}, \ldots, x_i^{(d)})$ for $i \in [nk]$ and denote $T_\ell = (x_1^{(\ell)}, \ldots, x_{nk}^{(\ell)})$ for $\ell \in [d]$. Then, also define the sign of a magic set as follows:

$$\mathrm{sgn}(T) := \mathrm{msgn}(T_1) \cdots \mathrm{msgn}(T_d).$$

Now, that it is shown that the signs of Latin hypercubes defined earlier are related as follows. This result will be useful in the next section.

Theorem 2.5.17. *Let $d > 2$ and $T \subseteq [k]^d$ be a magic set. For every partial Latin hypercube $C \in \mathcal{C}_d(k, T)$ holds*

$$\mathrm{sgn}_1(C)^{d-1}\mathrm{sgn}_2(C) \cdots \mathrm{sgn}_d(C)\,\mathrm{ssgn}(C) = \mathrm{sgn}(T).$$

Proof. In this proof consider all sums and equalities mod 2. The following notation is used

$$[A] := \begin{cases} 1, & A = true, \\ 0, & A = false. \end{cases}$$

Express signs as $\operatorname{sgn}(\sigma) = (-1)^{\#\text{ inversions of }\sigma}$ for a permutation σ.

For all $a = (a_1, \dots, a_d), b = (b_1, \dots, b_d) \in [k]^d$, define the functions $A_\ell : [k]^d \times [k]^d \to \{0,1\}$ for $\ell \in [d]$ and $B : [k]^d \times [k]^d \to \{0,1\}$ given by

$$A_\ell(a,b) := [a_\ell = b_\ell]\,[a < b]\,[C(a) > C(b)],$$
$$B(a,b) := [a_1 < b_1]\,[C(a) = C(b)]\sum_{i=2}^{d}[a_i > b_i].$$

where the relation $a < b$ corresponds to lexicographical order. Note that $A_\ell(a,b)$ expresses if the cells a and b lie in the same slice in the ℓ-th direction and form an inversion of a C-permutation. Similarly, B indicates contributions of the symbol sign. Then one can rewrite the sign functions as follows:

$$\operatorname{sgn}_\ell(C) = (-1)^{\sum_{a,b\in T} A_\ell(a,b)} \quad \text{and} \quad \operatorname{ssgn}(C) = (-1)^{\sum_{a,b\in T} B(a,b)}. \tag{2.15}$$

Define the indicators $\chi_i(*) : [k]^d \times [k]^d \to \{0,1\}$ given by $\chi_i(*)(a,b) = [a_i * b_i]$, where $*$ can be one of the binary relations $\{<, =, >, \cdot\}$ and $\cdot$ denotes the complete relation (identical 1). Similarly, define $\xi(*) : [k]^d \times [k]^d \to \{0,1\}$ given by $\xi(*)(a,b) = [C(a) * C(b)]$. The following fact is used: the functions $F(*) \in \{0,1\}$ satisfy the identity

$$F(=) + F(<) + F(>) = F(\cdot). \tag{2.16}$$

Denote $\chi_{i_1\dots i_\ell}(*) := \chi_{i_1}(*)\dots\chi_{i_\ell}(*)$. By exchanging the arguments a and b one can see that

$$\chi_{i_1\dots i_\ell}(<)\,\chi_{j_1\dots j_m}(>)\,\chi_{k_1\dots k_s}(=)\,\xi(<) = \chi_{i_1\dots i_\ell}(>)\,\chi_{j_1\dots j_m}(<)\,\chi_{k_1\dots k_s}(=)\,\xi(>) \tag{2.17}$$

for any pairwise distinct indices $i_1, \dots, i_\ell, j_1, \dots, j_m, k_1, \dots, k_s \in [d]$. Then the above functions A_ℓ and B can be rewritten as follows:

$$\begin{aligned}
A_\ell &= \left(\sum_{i=1, i\neq\ell}^{d} \chi_\ell(=)\chi_1(=)\dots\chi_{i-1}(=)\chi_i(<)\right)\xi(>),\\
B &= \sum_{i=2}^{d}\chi_1(<)\chi_i(>)\xi(=)\\
&= \sum_{i=2}^{d}\chi_1(<)\chi_i(>)(1 + \xi(<) + \xi(>))\\
&= \sum_{i=2}^{d}\chi_1(<)\chi_i(>) + \left(\sum_{i=2}^{d}\chi_1(<)\chi_i(>) + \chi_1(>)\chi_i(<)\right)\xi(>).
\end{aligned}$$

Denote $r = [d \text{ is odd}]$, then $\operatorname{sgn}_1(C)^{d-1} = \operatorname{sgn}_1(C)^{1+r}$. Using the fact $\chi_\ell(=)\chi_\ell(<) = 0$ obtain

$$\begin{aligned}
\sum_{\ell=1+r}^{d} A_\ell &= \sum_{\ell=1+r}^{d} \left(\sum_{i=1, i\neq\ell}^{d} \chi_\ell(=)\chi_{1\ldots i-1}(=)\chi_i(<) \right) \xi(>) \\
&= \sum_{\ell=1+r}^{d} \left(\sum_{i=1}^{d} \chi_\ell(=)\chi_{1\ldots i-1}(=)\chi_i(<) \right) \xi(>) \\
&= \underbrace{((d-r)\sum_{i=1}^{d}\chi_{1\ldots i-1}(=)\chi_i(<) +}_{=0 \text{ as } (d-r) \text{ is even}} \\
&+ \sum_{\ell=1+r}^{d}\sum_{i=1}^{d}\chi_{1\ldots i-1}(=)\chi_i(<)(\chi_\ell(<)+\chi_\ell(>)))\xi(>).
\end{aligned}$$

Further collecting the terms with the multiple $\chi_1(=)$ separately get

$$\begin{aligned}
\sum_{\ell=1+r}^{d} A_\ell = & \left(\sum_{\ell=2}^{d} \chi_1(<)(\chi_\ell(<)+\chi_\ell(>)) + (1-r)\chi_1(<) \right) \xi(>) \\
& + \chi_1(=)(\underbrace{\sum_{i=2}^{d}\sum_{\ell=i+1}^{d} \chi_{2\ldots i-1}(=)\chi_i(<)(\chi_\ell(<)+\chi_\ell(>)) +}_{=:L} \\
& + \underbrace{\sum_{i=2}^{d}\chi_{2\ldots i-1}(=)\chi_i(<)}_{=:R})\xi(>).
\end{aligned}$$

Let us collect the terms in R of the form $\chi_i(=)$ using the identity (2.16) consequently from right to left until the first non-dot relation appears, so that one can rewrite

$$R = \sum_{i=2}^{d}\chi_{2\ldots i-1}(=)\chi_i(<) = \sum_{i=2}^{d}\sum_{\ell=2}^{i-1}\chi_{2\ldots \ell-1}(=)(\chi_\ell(<)+\chi_\ell(>))\chi_i(<) + \sum_{i=2}^{d}\chi_i(<)$$

and therefore by identity (2.17) follows

$$\begin{aligned}
\chi_1(=)(L + R - \sum_{i=2}^{d}\chi_i(<))\xi(>) = \sum_{i=2}^{d}\sum_{\ell=i+1}^{d} & \chi_{1\ldots i-1}(=)(\chi_i(<)\chi_\ell(>)+ \\
& + \chi_i(>)\chi_\ell(<))\xi(>)
\end{aligned}$$

$$= \sum_{i=2}^{d} \sum_{\ell=i+1}^{d} \chi_{1\ldots i-1}(=)\chi_i(<)\chi_\ell(>)\left(\xi(<)+\xi(>)\right)$$
$$= \sum_{i=2}^{d} \sum_{\ell=i+1}^{d} \chi_{1\ldots i-1}(=)\chi_i(<)\chi_\ell(>)$$

where the latter equality is due to the fact $\chi_1(=)\xi(=) = 0$. Thus,

$$\begin{aligned}
\sum_{\ell=1+r}^{d} A_\ell + B &= \left(\sum_{\ell=2}^{d} \chi_1(<)(\chi_\ell(<)+\chi_\ell(>)) + (1-r)\chi_1(<)\right)\xi(>) \\
&\quad + \sum_{i=2}^{d} \chi_1(=)\chi_i(<)\xi(>) + \sum_{i=2}^{d}\sum_{\ell=i+1}^{d} \chi_{1\ldots i-1}(=)\chi_i(<)\chi_\ell(>) \\
&\quad + \sum_{i=2}^{d} \chi_1(<)\chi_i(>) + \left(\sum_{\ell=2}^{d} \chi_1(<)\chi_\ell(>) + \chi_1(>)\chi_\ell(<)\right)\xi(>) \\
&= \sum_{\ell=2}^{d} (\chi_1(<)+\chi_1(=)+\chi_1(>))\chi_\ell(<)\xi(>) + (1-r)\chi_1(<)\xi(>) \\
&\quad + \sum_{i=2}^{d}\sum_{\ell=i+1}^{d} \chi_{1\ldots i-1}(=)\chi_i(<)\chi_\ell(>) + \sum_{i=2}^{d} \chi_1(<)\chi_i(>) \\
&= \sum_{\ell=1+r}^{d} \chi_\ell(<)\xi(>) + \sum_{i=1}^{d}\sum_{\ell=i+1}^{d} \chi_{1\ldots i-1}(=)\chi_i(<)\chi_\ell(>).
\end{aligned}$$

Finally, compute the expressions when plugged into (2.15). The contribution of terms in the first sum in the latter expression can be computed explicitly using the fact that each slice of C contains each number in $[n]$ exactly once. i.e.

$$\sum_{a,b\in T} (\chi_\ell(<)\xi(>))(a,b) = k^{2d-2}\binom{k}{2}\binom{n}{2}.$$

There are $d-r$ equal contributions from this sum, and so they cancel out since $d-r$ is even. This shows that the signs product for C does not depend on its values at cells, but only depends on its type. Then the second sum turns into

$$\sum_{a,b\in T}\left(\sum_{i=1}^{d}\sum_{\ell=i+1}^{d} \chi_{1\ldots i-1}(=)\chi_i(<)\chi_\ell(>)\right)(a,b) =$$
$$\sum_{a,b\in T}\left(\sum_{\ell=1}^{d} \chi_\ell(>)\sum_{i=1}^{d} \chi_{1\ldots i-1}(=)\chi_i(<)\right)(a,b)$$

$$= \sum_{a,b \in T} \sum_{\ell=1}^{d} [a < b][a_\ell > b_\ell]$$

$$= \sum_{\ell=1}^{d} \mathrm{inv}(T_\ell).$$

Therefore, plugging these back in (2.15) get

$$\begin{aligned} \mathrm{sgn}_1(C)^{d-1}\mathrm{sgn}_2(C) \dots \mathrm{sgn}_d(C)\mathrm{ssgn}(C) = \\ = \mathrm{msgn}(T_1) \cdots \mathrm{msgn}(T_d) = \mathrm{sgn}(T) \end{aligned} \tag{2.18}$$

and the proof follows. □

Remark 2.5.18. Note that for odd d the formula (2.18) is asymmetric w.r.t. the first direction (which is due to the choice of lexicographical order of collecting C-permutations). In fact, one can show that similar formulas hold for any direction (by altering the choice of the order).

Corollary 2.5.19. *Let $d > 2$. For every Latin hypercube $C \in \mathcal{C}_k^{(d)}$ one have*

$$\mathrm{sgn}_1(C)^{d-1}\mathrm{sgn}_2(C) \cdots \mathrm{sgn}_d(C)\,\mathrm{ssgn}(C) = (-1)^{\lfloor \frac{d}{2} \rfloor \lfloor \frac{k}{2} \rfloor k}.$$

Proof. By Theorem 2.5.17 for $T = [k]^d$ one have

$$\mathrm{sgn}_1(C)^{d-1}\mathrm{sgn}_2(C) \cdots \mathrm{sgn}_d(C) = \mathrm{sgn}([k]^d).$$

Note that $T_\ell = (1^{k^{d-\ell}}, \dots, k^{k^{d-\ell}})^{k^{\ell-1}}$ for $\ell \in [d]$ (where denote $x^n = (x, \dots, x)$ as a sequence of n elements and $(a_1, \dots, a_n)^m = (a_1, \dots, a_n, \dots, a_1, \dots, a_n)$ is a sequence of nm elements). Then

$$\mathrm{inv}(T_\ell) = \binom{k^{\ell-1}}{2}\binom{k}{2}k^{2d-2\ell}.$$

If ℓ is even, then $\binom{k^{\ell-1}}{2} \equiv \binom{k}{2} \equiv \lfloor \frac{k}{2} \rfloor \pmod 2$ and if ℓ is odd, then $\binom{k^{\ell-1}}{2} \equiv 0 \pmod 2$. Hence,

$$\sum_{\ell=1}^{d} \mathrm{inv}(T_\ell) \equiv \lfloor d/2 \rfloor \lfloor k/2 \rfloor k \pmod 2$$

and the statement follows. □

Remark 2.5.20. Our proof is in the spirit of Janssen's proof [55] showing that for every Latin square L of length k

$$\mathrm{sgn}_1(L)\,\mathrm{sgn}_2(L)\,\mathrm{ssgn}(L) = (-1)^{\lfloor \frac{k}{2} \rfloor}.$$

2.5.3 Highest weight spaces

In this section the following result is proved.

Theorem 2.5.21. *Let $d \geqslant 3$ be odd. If $AT_d(k) \neq 0$ for even k, then $g_d(n,k) > 0$ for all $n \leqslant k^{d-1}$.*

In particular, the condition $AT_3(k) \neq 0$ implies $g_3(n,k) > 0$ for all $n \leqslant k^2$ and hence by Theorem 2.4.5 (iii) also establishes the positivity $g_d(n,k) > 0$ for all $n \leqslant k^{d-1}$. The proof is based on description of highest weight vectors.

a) Highest weight vectors

The group $\Gamma = \mathrm{GL}(k)^{\times d}$ acts on the anti-symmetric space $\bigwedge^m(\mathbb{C}^k)^{\otimes d}$ componentwise. A vector $v \in \bigwedge^m(\mathbb{C}^k)^{\otimes d}$ is called a *weight vector* if it is rescaled by the action of diagonal matrices, i.e.

$$(\mathrm{diag}(a_1^{(1)}, \ldots, a_k^{(1)}), \ldots, \mathrm{diag}(a_1^{(d)}, \ldots, a_k^{(d)})) \cdot v = (a^{(1)})^{\lambda^{(1)}} \ldots (a^{(d)})^{\lambda^{(d)}} v,$$

where $x^\alpha = x_1^{\alpha_1} \ldots x_k^{\alpha_k}$; then $(\lambda^{(1)}, \ldots, \lambda^{(d)})$ is the *weight* of v. The weight vectors of the same weight form a *weight space*. Any representation of Γ can be decomposed into a direct sum of weight spaces. Further, an explicit weight space decomposition from [56] is described.

The elements of $[k]^d$ are always viewed ordered lexicographically. Let $P \subseteq [k]^d$. Denote by $s_\ell(P,i)$ the number of elements in P whose ℓ-th coordinate is i, i.e. the number of points of i-th slice in ℓ-th direction, and $s_\ell(P) := (s_\ell(P,1), \ldots, s_\ell(P,k))$ are the vectors of *marginals*. For $P = \{(x_i^{(1)}, \ldots, x_i^{(d)}) : 1 \leqslant i \leqslant m\}$ associate the vector

$$\psi_P := \bigwedge_{i=1}^{m} e_{x_i^{(1)}} \otimes \ldots \otimes e_{x_i^{(d)}}$$

where $\{e_i\}_{i=1}^k$ is the standard basis of $\mathbb{C}^k$. The vectors $\{\psi_P\}$ over all possible $P \subseteq [k]^d$ of cardinality m form a basis of $\bigwedge^m(\mathbb{C}^k)^{\otimes d}$. Moreover, it is straightforward to verify that ψ_P is a weight vector of weight $(s_1(P), \ldots, s_d(P))$ for the action of Γ.

Let $V(\lambda)$ be the irreducible representation of $\mathrm{GL}(k)$ indexed by partition λ (known as Weyl module).

Lemma 2.5.22 (cf. [56, Lemma 2.1])**.** *Let $\lambda^{(1)}, \ldots, \lambda^{(d)}$ be partitions of size m whose first parts are at most k. The Kronecker coefficient $g(\lambda^{(1)}, \ldots, \lambda^{(d)})$ is equal to the multiplicity of the irreducible representation $V((\lambda^{(1)})') \otimes \ldots \otimes V((\lambda^{(d)})')$ in the anti-symmetric space $\bigwedge^m(\mathbb{C}^k)^{\otimes d}$, where λ' is the conjugate of λ.*

Each irreducible representation $V((\lambda^{(1)})') \otimes \ldots \otimes V((\lambda^{(d)})')$ contains a unique one-dimensional space of weight $(\lambda^{(1)}, \ldots, \lambda^{(d)})$ called the *highest weight space*, and so $g(\lambda^{(1)}, \ldots, \lambda^{(d)})$ is the dimension of the corresponding highest weight space, i.e.

$$g(\boldsymbol{\lambda}) = \dim \mathrm{HWV}_{\boldsymbol{\lambda}'} \bigwedge^m V. \tag{2.19}$$

The *highest weight vectors*, which are elements of the highest weight space, can be characterized by the action of the Lie algebra $\mathfrak{g}$ of Γ. Let $A \in \mathfrak{g}$ and I be the unit of Γ. Then $\epsilon A + I \in \Gamma$ for sufficiently small ϵ and the action of $A \in \mathfrak{g}$ on v is defined by $Av := \lim_{\epsilon \to 0}((\epsilon A + I)v - v)\epsilon^{-1}$. Let $E_{i,j}$ be the $k \times k$ matrix with a single 1 at entry (i, j) and zero elsewhere. For $i < j$ the operator $v \mapsto (E_{i,j}, 0, 0, \ldots)v$ is called a *raising operator* in direction 1 (which is defined similarly for other directions). Then a weight vector vanishing by all raising operators is a highest weight vector.

The case $\lambda^{(1)} = \ldots = \lambda^{(d)} = n \times k$ is of largest interest, motivated by the questions in previous sections. Hence one needs to consider the weight spaces of weight $(k \times n)^d$. That is, the span of vectors ψ_P where P runs over sets with marginals $s_\ell(P) = n \times k$ such that P has n elements in each of k slices in any direction, i.e. P are magic sets (defined in the previous section). Define the following subspaces of $\bigwedge^{nk}(\mathbb{C}^k)^{\otimes d}$:

$$\mathcal{B}_{d,k}(n) := \operatorname{span}\{\psi_P \mid P \subseteq [k]^d \text{ is a magic set of cardinality } kn\}$$

for $n = 0, \ldots, k^{d-1}$. Each $\mathcal{B}_{d,k}(n)$ is the weight space of weight $(k \times n)^d$. In particular, $\mathcal{B}_{d,k}(1)$ is a vector space with the basis indexed by permutation hypermatrices and $\mathcal{B}_{d,k}(k^{d-1})$ is a one-dimensional vector space with a single basis vector $\psi_{[k]^d}$.

The action of raising operators on the highest weight subspace of the space $\mathcal{B}_{d,k}(n)$ can be described combinatorially. The generators (in the direction 1) are of the form $E^{(1)}_{i,i+1} := (E_{i,i+1}, 0, 0, \ldots)$. The action is linear and defined on basis vectors as follows:

$$E^{(1)}_{i,i+1}\psi_P = \sum_{j=1}^{n} (-1)^{j+1}[x_j^{(1)} = i+1]\, e_i \otimes e_{x_j^{(2)}} \otimes \ldots \otimes e_{x_j^{(d)}} \wedge \bigwedge_{\ell=1,\ell\neq j}^{n} e_{x_\ell^{(1)}} \otimes \ldots \otimes e_{x_\ell^{(d)}}$$

for any $P = \{(x_1^{(1)}, \ldots, x_1^{(d)}), \ldots, (x_n^{(1)}, \ldots, x_n^{(d)})\}$ and $i \in [k-1]$, where $[A] = 1$ if A is true and 0 otherwise. In other words, for each $p \in P$ with the first coordinate $i+1$, replace p with the point $p' = p - (1, 0, 0, \ldots)$ without changing the order of elements in P. Denote the new (ordered) set by $P - p + p'$. Then the action of the raising operator can also be written as follows:

$$E^{(1)}_{i,i+1}\psi_P = \sum_{p=(i+1,\ldots)\in P} \psi_{P-p+p'}.$$

Note that $P - p + p'$ has n points in each slice except the slices i and $i+1$ in the direction 1 where it has $n+1$ and $n-1$ elements, respectively. The action or raising operators in other directions is defined similarly.

Combine all raising operators into a single operator E given by $E := \sum_{j=1}^{d} \sum_{i=1}^{k-1} E_{i,i+1}^{(j)}$. Since the images of $E_{i,i+1}^{(j)}$ do not intersect, then:

$$v \in \mathcal{B}_{d,k}(n) \text{ is highest a weight vector} \iff E(v) = 0.$$

Alternatively, $v \in \ker E$. Thus, $\dim \ker E|_{\mathcal{B}_{d,k}(n)} = g_d(n,k)$.

Lemma 2.5.23. *Let $\alpha \in \mathcal{B}_{d,k}(n)$ and $\beta \in \mathcal{B}_{d,k}(m)$ be highest weight vectors and assume $\alpha \wedge \beta \neq 0$. Then $\alpha \wedge \beta \in \mathcal{B}_{d,k}(n+m)$ is also a highest weight vector.*

Proof. Checking the action of raising operators E then

$$E(\alpha \wedge \beta) = E(\alpha) \wedge \beta + \alpha \wedge E(\beta) = 0$$

and the statement follows. □

b) A hyperdeterminantal form

Define the following special *hyperdeterminantal* element of the space $\mathcal{B}_{d,k}(1)$:

$$\omega := \sum_{\pi_2,\ldots,\pi_d \in S_k} \operatorname{sgn}(\pi_2 \ldots \pi_d) \bigwedge_{i=1}^{k} e_i \otimes e_{\pi_2(i)} \otimes \ldots \otimes e_{\pi_d(i)} \in \mathcal{B}_{d,k}(1). \tag{2.20}$$

Write

$$\omega^n := \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}} \in \mathcal{B}_{d,k}(n).$$

Theorem 2.5.24. *Let $d \geqslant 3$ be odd. Then:*

(i) ω is a unique (up to a scalar) highest weight vector in $\mathcal{B}_{d,k}(1)$ (and is $\mathrm{SL}(k)^{\times d}$-invariant).

(ii) For k even and $n \leqslant k^{d-1}$,

$$\omega^n = \pm \sum_T \operatorname{sgn}(T) AT_d(k,T)\, \psi_T,$$

where the sum is over all magic sets $T \subseteq [k]^d$ of cardinality $|T| = nk$. In particular, for $T = [k]^d$ obtain

$$\omega^{k^{d-1}} = \pm AT_d(k)\, \psi_{[k]^d},$$

(iii) For k odd, $\omega^2 = 0$.

Proof. Let $\mathcal{S} = \{\pi = (\pi_1, \ldots, \pi_d) : \pi \in (S_k)^d,\ \pi_1 = \mathrm{id}\}$. For $\pi = (\pi_1, \ldots, \pi_d) \in \mathcal{S}$ denote $\operatorname{sgn}(\pi) := \operatorname{sgn}(\pi_1 \cdots \pi_d)$ and $\pi(i) = (\pi_1(i), \ldots, \pi_d(i)) \in [k]^d$. Let us also denote $e_{\pi(i)} := e_{\pi_1(i)} \otimes \cdots \otimes e_{\pi_d(i)}$ and $e_\pi := \bigwedge_{i=1}^{k} e_{\pi(i)}$.

(i) Firstly note that $g_d(1,k) = 1$ and hence there is only one highest weight vector in $\mathcal{B}_{d,k}(1)$. Now one needs to verify that the action of raising operators vanishes ω. Let us check it on the generators of the form $E^{(1)}_{i,i+1} = (E_{i,i+1}, 0, 0, \ldots)$.

For $\pi \in \mathcal{S}$ let $L_\pi = \{\pi(1), \ldots, \pi(i-1)\}$, $a_\pi = \pi(i)$, $b_\pi = \pi(i+1)$ and $R_\pi = \{\pi(i+2), \ldots, \pi(k)\}$. Applying the raising operator to ω get

$$(E_{i,i+1}, 0, \ldots, 0)\, \omega = \sum_{\pi \in \mathcal{S}} \operatorname{sgn}(\pi)\, \psi_{\{L_\pi, a_\pi, b'_\pi, R_\pi\}} \tag{2.21}$$

where the set considered as ordered and for $x \in [k]^d$ denote $x' = x - (1, 0, \ldots, 0)$. Note that the cells a_π and b'_π both lie in the i-th slice of the first direction in $[k]^d$, and so the last sum contains only wedge products ψ_S indexed by (ordered) sets S having two elements in the slice i and no elements in the slice $i+1$. Let us fix a set $P = L \cup \{a', b'\} \cup R$ where L and R are (lexicographically ordered) sets of cells in the slices $1, \ldots, i-1$ and $i+2, \ldots, k$ respectively, and $a < b$ are cells from the slice $i+1$. Consider the coefficient c_P at the vector ψ_P in the sum (2.21). There are exactly two tuples of permutations π^1 and π^2 in $\mathcal{S}$ which contribute to the coefficient at ψ_P, defined as $L_{\pi^1} = L_{\pi^2} = L$, $R_{\pi^1} = R_{\pi^2} = R$ and

$$\begin{aligned} a_{\pi^1} &= \pi^1(i) = a', \quad & b_{\pi^1} &= \pi^1(i+1) = b, \\ a_{\pi^2} &= \pi^2(i) = b', \quad & b_{\pi^2} &= \pi^2(i+1) = a. \end{aligned}$$

First, observe that $c_P = \operatorname{sgn}(\pi^1) - \operatorname{sgn}(\pi^2)$, since

$$c_P\, \psi_P = \operatorname{sgn}(\pi^1)\, \psi_{\{L, a', b', R\}} + \operatorname{sgn}(\pi^2)\, \psi_{\{L, b', a', R\}} = (\operatorname{sgn}(\pi^1) - \operatorname{sgn}(\pi^2))\, \psi_P.$$

Second, let us check that $\operatorname{sgn}(\pi^1) = \operatorname{sgn}(\pi^2)$. Indeed,

$$\pi^1 = (\mathrm{id}, \pi^1_2, \ldots, \pi^1_d) = (\mathrm{id}, (i, i+1) \cdot \pi^2_2, \ldots, (i, i+1) \cdot \pi^2_d),$$

where $(i, i+1) \in S_k$ is a transposition, and hence, $\operatorname{sgn}(\pi^1) = (-1)^{d-1} \operatorname{sgn}(\pi^2) = \operatorname{sgn}(\pi^2)$. Therefore, $c_P = 0$ and since P was chosen arbitrarily, obtain that $(E_{i,i+1}, 0, \ldots, 0)\, \omega = 0$. The same argument works in other directions by rewriting ω as follows. For odd d the choice of the fixed first direction $\pi_1 = id$ can be changed, namely, for any $\ell \in [d]$:

$$\begin{aligned} \omega &= \sum_{\pi_2, \ldots, \pi_d \in S_k} \operatorname{sgn}(\pi_2 \ldots \pi_d) \bigwedge_{i=1}^{k} e_i \otimes e_{\pi_2(i)} \otimes \ldots \otimes e_{\pi_d(i)} \\ &= \sum_{\pi_2, \ldots, \pi_d \in S_k} \operatorname{sgn}(\pi_\ell)^{d-1} \operatorname{sgn}(\pi_2 \ldots \widehat{\pi}_\ell \ldots \pi_d) \bigwedge_{i=1}^{k} e_{\pi_\ell^{-1}(i)} \otimes \ldots \otimes e_i \otimes \ldots \otimes e_{\pi_\ell^{-1} \pi_d(i)} \end{aligned}$$

$$= \sum_{\pi_1,\ldots,\pi_{\ell-1},\pi_{\ell+1},\ldots,\pi_d \in S_k} \operatorname{sgn}(\pi_1 \ldots \widehat{\pi}_\ell \ldots \pi_d) \bigwedge_{i=1}^{k} e_{\pi_1(i)} \otimes \ldots \otimes e_i \otimes \ldots \otimes e_{\pi_d(i)}$$

(where as usual $\widehat{\cdot}$ denotes the absence in the sequence).

(ii) For each $\pi \in \mathcal{S}$ the set $d(\pi) := \{\pi(1), \ldots, \pi(k)\}$ is a diagonal of $[k]^d$. So $\omega = \sum_{\pi \in \mathcal{S}} \operatorname{sgn}(\pi)\, e_\pi$ where each term with index π in the sum corresponds to the diagonal $d(\pi)$. Let us denote

$$M_{d,k}(n) := \left\{T \subseteq [k]^d : T \text{ is a magic set of cardinality } nk\right\}.$$

Consider the expansion

$$\begin{aligned}
\omega^n &= \left(\sum_{\pi \in \mathcal{S}} \operatorname{sgn}(\pi) e_\pi\right)^{\wedge n} \\
&= \sum_{\pi^1,\ldots,\pi^n \in \mathcal{S}} \operatorname{sgn}(\pi^1) \cdots \operatorname{sgn}(\pi^n)\, e_{\pi^1} \wedge \ldots \wedge e_{\pi^n} \\
&= \sum_{T \in M_{d,k}(n)} \sum_{\pi^1,\ldots,\pi^n \in \mathcal{S}:\ \cup_{i=1}^{k} d(\pi^i) = T} \operatorname{sgn}(\pi^1) \cdots \operatorname{sgn}(\pi^n)\, e_{\pi^1} \wedge \ldots \wedge e_{\pi^n}
\end{aligned}$$

since if for some $i < j$ one have $d(\pi^i) \cap d(\pi^j) \neq \varnothing$ then a corresponding term vanishes in the above expansion due to the property of the wedge product; hence it is assumed that $d(\pi^1), \ldots, d(\pi^n)$ form a partition of T into diagonals. Rewrite this sum as follows:

$$\begin{aligned}
\omega^n = \sum_{T \in M_{d,k}(n)} \sum_{\pi^1,\ldots,\pi^n \in \mathcal{S}:\ \cup_{i=1}^{n} d(\pi^i) = T} \operatorname{sgn}(\pi^1) \cdots \operatorname{sgn}(\pi^n)\, & e_{\pi^1(1)} \wedge \cdots \wedge e_{\pi^1(k)} \wedge \\
& e_{\pi^2(1)} \wedge \cdots \wedge e_{\pi^2(k)} \wedge \\
& \ldots \\
& e_{\pi^n(1)} \wedge \cdots \wedge e_{\pi^n(k)}.
\end{aligned}$$

Let us fix $T \in M_{d,k}(n)$ and $\pi^1, \ldots, \pi^n \in \mathcal{S}$ such that $\bigcup_{i=1}^n d(\pi^i) = T$ is a set partition. Then construct a partial Latin hypercube C as follows: for each $i \in [n]$ set $C(d(\pi^i)) = i$, i.e. each cell of the diagonal $d(\pi^i)$ has value i in C, and 0 otherwise. Then by definition

$$\operatorname{ssgn}(C) = \operatorname{sgn}(\pi^1) \cdots \operatorname{sgn}(\pi^n).$$

Let σ_j be the permutation formed by values of C in j-th slice of the first direction collected in lexicographical order, then

$$\operatorname{sgn}_1(C) = \operatorname{sgn}(\sigma_1) \cdots \operatorname{sgn}(\sigma_k).$$

Since the union of the diagonals covers T, the corresponding vector term is up to a sign equal to ψ_T. Let us describe the canonical order of wedge multiples. Note that for a fixed $j \in [k]$ the cells $\pi^1(j), \ldots, \pi^n(j) \in [k]^d$ form the full j-th slice of T in the first direction. Let us reorder the wedge multiples $e_{\pi^1(j)}, \ldots, e_{\pi^n(j)}$ lexicographically by indices, this is achieved via the permutation σ_j^{-1}. Let $a_{j,1}, \ldots, a_{j,n} \in T$ be the elements of T in j-th slice, written in lexicographical order. Then

$$e_{\pi^1(j)} \wedge \ldots \wedge e_{\pi^n(j)} = \operatorname{sgn}(\sigma_j)\, e_{a_{j,1}} \wedge e_{a_{j,2}} \wedge \ldots \wedge e_{a_{j,n}}.$$

Denote the vector term

$$\begin{aligned}\phi_T := \; & e_{a_{1,1}} \wedge \ldots \wedge e_{a_{k,1}} \wedge \\ & e_{a_{2,n}} \wedge \ldots \wedge e_{a_{k,2}} \wedge \\ & \ldots \\ & e_{a_{1,n}} \wedge \ldots \wedge e_{a_{k,n}}.\end{aligned}$$

Then obtain

$$\omega^n = \sum_{T \in M_{d,k}(n)} \left(\sum_{C \in \mathcal{C}_d(k,T)} \operatorname{ssgn}(C) \operatorname{sgn}_1(C) \right) \cdot \; \phi_T.$$

By Theorem 2.5.17 for odd d holds

$$\operatorname{sgn}_1(C) \operatorname{ssgn}(C) = \operatorname{sgn}_1(C) \operatorname{sgn}_2(C) \cdots \operatorname{sgn}_d(C) \operatorname{sgn}(T) = \operatorname{sgn}(C) \operatorname{sgn}(T).$$

Note that $\phi_T = (-1)^\ell \psi_T$ for any T, where ℓ is a constant which does not depend on T. Therefore,

$$\omega^n = \sum_{T \in M_{d,k}(n)} \operatorname{sgn}(T)\, AT_d(k,T)\, \phi_T = \pm \sum_{T \in M_{d,k}(n)} \operatorname{sgn}(T)\, AT_d(k,T)\, \psi_T.$$

(iii) Let us now show that if k is odd, then $\omega^2 = 0$. Choose an arbitrary total order $<$ on $\mathcal{S}$. A straightforward calculation shows that

$$\omega^2 = \sum_{\pi^1, \pi^2 \in \mathcal{S}, \pi^1 < \pi^2} \operatorname{sgn}(\pi^1) \operatorname{sgn}(\pi^2)\, (e_{\pi^1} \wedge e_{\pi^2} + e_{\pi^2} \wedge e_{\pi^1}) = 0$$

where it is used that $e_{\pi^2} \wedge e_{\pi^1} = (-1)^k e_{\pi^1} \wedge e_{\pi^2} = -e_{\pi^1} \wedge e_{\pi^2}$. □

Remark 2.5.25. As pointed out by the referee, the part (i) can also be shown as follows. It is known (see e.g. [37, Sec. 4.2]) that the space of highest weight vectors of $\otimes^{dk}\mathbb{C}^k$ of weight $\lambda^d = (1 \times k)^d$ is obtained by applying the Young symmetrizer P_{λ^d} to $\otimes^{dk}\mathbb{C}^k$ given by

$$P_\lambda = \frac{1}{k!^d} \sum_{\pi_1,\ldots,\pi_d \in S_k} \operatorname{sgn}(\pi_1)\cdots\operatorname{sgn}(\pi_d)(\pi_1,\ldots,\pi_d)$$

which maps $(e_1 \otimes \ldots \otimes e_k)^{\otimes d}$ to $k!\omega$. Here the action of $(S_k)^d$ on $\otimes^{dk}\mathbb{C}^k$ is considered, such that each group copy acts independently on k tensor factors by permuting them.

Corollary 2.5.26. *Let $d \geqslant 3$ be odd and $2 \leqslant n \leqslant k^{d-1}$. The following statements are equivalent:*

(1) $\omega^n \neq 0$

(2) $AT_d(k,T) = \Delta_T(I_n) \neq 0$ for some magic set $T \subseteq [k]^d$ of cardinality nk (or a $d \times nk$ balanced table)

(3) $\omega^i \in \mathcal{B}_{d,k}(i)$ is a highest weight vector for all $1 \leqslant i \leqslant n$.

Proof. (1) $\iff$ (2) follows directly from the expansion in (ii) since the vectors ψ_T are linearly independent.

The condition $\omega^n \neq 0$ is equivalent to $\omega^i \neq 0$ for all $i \leqslant n$. Since ω is a highest weight vector, by Lemma 2.5.23, if $\omega^i \neq 0$ then it is a highest weight vector for all $i \leqslant n$. □

Corollary 2.5.27. *Let $d \geqslant 3$ be odd and assume $AT_d(k) \neq 0$ for even k. Then $\omega^n \in \mathcal{B}_{d,k}(n)$ is a highest weight vector for all $1 \leqslant n \leqslant k^{d-1}$. In particular, $g_d(n,k) > 0$ for all $n \leqslant k^{d-1}$.*

This establishes Theorem 2.5.21.

Also, the following weaker statement conditional on the Alon–Tarsi conjecture $AT(k) = AT_2(k) \neq 0$ for even k.

Corollary 2.5.28. *Let $d \geqslant 3$ be odd. Assume $AT(k) \neq 0$ for even k. Then $g_d(n,k) > 0$ for all $n \leqslant k$.*

Proof. Let us show that $\omega^k \neq 0$ and hence ω^n is a highest weight vector for all $n \leqslant k$. By Proposition 2.5.3 for some $d \times k^2$ balanced table $\Delta_T(I_k) \neq 0$ is equivalent to $AT(k) \neq 0$. Hence, $\omega^k \neq 0$ by the results above. □

Remark 2.5.29. Hence, $AT(k) \neq 0$ also implies that $g_d(n,k) = g_d(k^{d-1}-n,k) > 0$ for $n \leqslant k$ and as noted before one gets $\delta_d(k^{d-1}-n) = (k^{d-1}-n)k$ for all $n \leqslant k$. Of course, $AT_3(k) \neq 0$ gives much more information on the degree sequence $\delta_d(n)$ as discussed in the introduction.

Remark 2.5.30. For $d = 3$ analogous statements on positivity of Kronecker coefficients as in the last corollary can also be obtained from results in [57], see [58].

Remark 2.5.31. For $n > 1$ let $k_d(n)$ be minimal number k for which there is an invariant Δ_T of degree kn such that $\Delta_T(I_n) \neq 0$. Note that $k_d(n) \geqslant \delta_d(n)/n$ and $k_d(n)$ is even, cf. Corollary 2.5.13 and Remark 2.5.14. By Corollary 2.5.26 if $\omega^n \neq 0$ then $\omega^{n-1} \neq 0$, and hence if there is an invariant Δ_T of degree nk with $\Delta_T(I_n) \neq 0$, then there is also an invariant $\Delta_{T'}$ of degree $(n-1)k$ with $\Delta_{T'}(I_{n-1}) \neq 0$. This implies the monotonicity $k_d(n) \geqslant k_d(n-1)$. Note that such property fails for $\delta_3(n)/n$, e.g. $\delta_3(8)/8 = 3 < \delta_3(7)/7 = 4$.

Remark 2.5.32. The dual of the element ω is actually Cayley's first hyperdeterminant, cf. Example 2.3.8. Let d be odd. For any $x_1, \dots, x_k \in (\mathbb{C}^k)^{\otimes d}$ let $X \in (\mathbb{C}^k)^{\otimes(d+1)}$ be tensor formed via concatenation of x_i by slices, i.e. $X = \sum_{i=1}^k e_i \otimes x_i$. Then

$$\omega^*(x_1, \dots, x_k) = \frac{1}{k!} \sum_{\sigma_1, \dots, \sigma_d \in S_k} \operatorname{sgn}(\sigma_1 \cdots \sigma_d) \prod_{i=1}^k X_{i, \sigma_1(i), \dots, \sigma_d(i)}.$$

The element $(\omega^*)^{\wedge n}$ can be regarded as a multi-linear skew-symmetric nk-form on the space $(\mathbb{C}^k)^{\otimes d}$ for $n \leqslant k^{d-1}$, as well as a polynomial function on $(\mathbb{C}^k)^{d+1}$ which is multi-linear and skew-symmetric in slices. This nk-form can be calculated as follows:

$$(\omega^n)^*(x_1, \dots, x_{nk}) = \sum_{I_1, \dots, I_n} \varepsilon_{I_1 \dots I_n} \, \omega^*(x_{i_{11}}, \dots, x_{i_{1k}}) \cdots \omega^*(x_{i_{n1}}, \dots, x_{i_{nk}})$$

where the sum is over subsets $I_1, \dots, I_n \subseteq [nk]$ with $I_j = \{i_{j1} < \dots < i_{jk}\}$ and $I_1 \cup \dots \cup I_n = [nk]$, and $\varepsilon_{I_1, \dots, I_n} = \operatorname{sgn}(i_{11}, \dots, i_{1k}, \dots, i_{n1}, \dots, i_{nk})$.

c) Unconditional Kronecker positivity

Finally, using the conditional statement in Corollary 2.5.28 one can obtain the following unconditional result.

Theorem 2.5.33. *Let $d \geqslant 3$ be odd and k be even. Then $g_d(n, k) > 0$ for all $n \leqslant \sqrt{k}/2 - 1$.*

Proof. It is known that for every prime $p \geqslant 3$ holds $AT(p \pm 1) \neq 0$ (see [53, 54]) and therefore from Corollary 2.5.28 $g_d(n, p \pm 1) > 0$ for all $n \leqslant p \pm 1$. Hence, using the semigroup property of Kronecker coefficients one gets that $g_d(n, a(p-1) + b(p+1)) > 0$ for any $a, b \in \mathbb{N}$ and all $n \leqslant p - 1$.

For $k \leqslant 8$ the statement can be checked computationally. For $k \geqslant 10$, choose a prime $p \in [\sqrt{k}/2, \sqrt{k}]$ and write $k = \ell p + r$ so that $r \in [0, p-1]$ and $\ell - r$ is even; note that $\ell - r > 0$ since $\ell + 1 > k/p \geqslant p > r$. Hence, k can be presented as follows:

$$k = \ell p + r = \frac{\ell + r}{2}(p+1) + \frac{\ell - r}{2}(p-1).$$

Therefore, using the semigroup property as noted above obtain that $g_d(n,k) > 0$ for all $n \leqslant p-1$ and the result follows. $\square$

Remark 2.5.34. Since it is known that for any sufficiently large k there is a prime in the interval $[\sqrt{k} - o(\sqrt{k}), \sqrt{k}]$, one can get a better bound that $g_d(n,k) > 0$ for all $n \leqslant \sqrt{k} - o(\sqrt{k})$.

Corollary 2.5.35. *Let $d \geqslant 3$ be odd and k be even. Then $\delta_d(n) = nk$ for all $k^{d-1} - \sqrt{k}/2 + 1 \leqslant n \leqslant k^{d-1}$.*

This completes part of Theorem 2.3.2.

Now, another construction that shows positivty of Kronecker coefficients is provided.

Definition 2.5.36. The *direct sum* $\oplus$ of magic sets T_1 and T_2 both of magnitude n in the cubes $[k_1]^d$ and $[k_2]^d$ respectively, is a magic set given by

$$T_1 \oplus T_2 = T_1 \cup \{(a_1 + k_1, \ldots, a_d + k_1) : (a_1, \ldots, a_d) \in T_2\}$$

of magnitude n in the cube $[k_1 + k_2]^d$.

A useful property of this operation is the following.

Proposition 2.5.37. *For $T_1 \in B_{d,k_1}(n)$ and $T_2 \in B_{d,k_2}(n)$ holds*

$$\mathrm{AT}(T_1 \oplus T_2) = \mathrm{AT}(T_1) \cdot \mathrm{AT}(T_1).$$

Proof. Any partial Latin hypercube $L \in \mathcal{C}_d(T_1 \oplus T_2)$ decomposes into a pair of Latin hypercubes $(L_1, L_2) \in \mathcal{C}_d(T_1) \times \mathcal{C}_d(T_2)$ with $\mathrm{sgn}(L) = \mathrm{sgn}(L_1)\,\mathrm{sgn}(L_2)$, since each slice of $[k_1 + k_2]^d$ intersects $T_1 \oplus T_2$ either in T_1 or in T_2. $\square$

Theorem 2.5.38. *For odd $d \geqslant 3$ and even k holds $\omega^{2^{d-1}} \neq 0$.*

Proof. Let $T = ([2]^d)^{\oplus \frac{k}{2}} \in B_{d,k}(n)$. Then from Theorem 2.5.24 the coefficient of $\omega^{2^{d-1}}$ at e_T up to a sign is equal to

$$\pm \mathrm{AT}(T) = \pm \mathrm{AT}_d(2)^{k/2} \neq 0$$

since $\mathrm{AT}([2]^d) = \mathrm{AT}_d(2) \neq 0$. $\square$

Corollary 2.5.39. *For odd $d \geqslant 3$ and even k holds $g_d(n,k) > 0$ for all $n \leqslant 2^{d-1}$.*

Proof. Since $\omega^{2^{d-1}} \neq 0$ then $\omega^n \neq 0$ for all $n \leqslant 2^{d-1}$. But

$$g_d(n,k) = \dim \mathrm{HWV}_{(k \times n)^d} \bigwedge^{nk} (\mathbb{C}^k)^{\otimes d}$$

and ω^n is an element of latter space. $\square$

Remark 2.5.40. For fixed d_0 and sufficiently large d by the above corollary one finds that $(\omega_{d,k})^{k^{d_0-1}} \neq 0$ such that one of the coefficients up to a sign is equal to $\mathrm{AT}_{d_0}(k)$.

Remark 2.5.41. By analogous reasoning, for $n \leqslant 2^{d-1}$ one can construct $\mathrm{SL}(n)^{\times d}$-invariant polynomial $\Delta_T \in \mathbb{C}[(\mathbb{C}^n)^{\otimes d}]_{2n}$ of minimal possible degree $2n$ (where $T \in B_{d,2}(n)$), such that $\Delta_T(I_n) = \mathrm{AT}(T) \neq 0$ (cf. [59]), where $I_n = \sum_{i=1}^{n}(e_i)^{\otimes d}$ is the unit tensor (which is semistable for action of $\mathrm{SL}(n)^{\times d}$).

Corollary 2.5.42. *For even k and odd $d \geqslant 1 + \log_2 k$ one have $\omega^k \neq 0$.*

Remark 2.5.43. From [59] it is known that one of the coefficients in expansion of ω^k is equal to $\mathrm{AT}_2(k)$. Taking d sufficiently large one can also obtain that $\omega^k \neq 0$. One may consider relations between the coefficients in expansion of ω^k towards the Alon–Tarsi conjecture.

2.5.4 Conclusion

The study of fundamental invariants and Latin hypercubes has provided significant insights into the algebraic structure and combinatorial properties of tensor invariants in higher-dimensional spaces. The explicit construction of balanced tables T and $\widetilde{T}$, along with their corresponding invariants $F_{d,k}$ and $\widetilde{F}_{d,k}$, establishes the uniqueness of these fundamental invariants. This uniqueness is crucial for understanding the graded structure of the ring of G-invariant polynomials.

Furthermore, the investigation into Latin hypercubes and their relation to the Alon-Tarsi conjecture has shown that the nonzero evaluation of the invariant Δ_T at the unit tensor I_k is equivalent to the nonzero value of the Alon-Tarsi number $AT(k)$. This equivalence links combinatorial conjectures with algebraic invariants, providing a deeper understanding of the interplay between these fields.

The proof that $\Delta_T(I_k) \neq 0$ implies $AT(k) \neq 0$ for even k highlights the potential of using tensor invariants to tackle long-standing combinatorial problems. Additionally, the introduction of magic sets and partial Latin hypercubes enriches the framework for analyzing tensor invariants by connecting them to well-studied combinatorial objects.

In summary, the results presented in this section offer a comprehensive framework for the study of tensor invariants, bridging the gap between combinatorial and algebraic perspectives. This foundational work not only advances the theoretical understanding of tensor invariants but also paves the way for future research into their applications in quantum information theory and other areas of mathematics and physics.

2.6 Unimodality of Kronecker coefficients

Content of this section on unimodality is given in [60].

A sequence $\{x_n\}_{n=a,\ldots,b}$ is *unimodal* if $x_a \leqslant \ldots \leqslant x_m \geqslant \ldots \geqslant x_b$ for some $m \in [a, b]$, and it is *symmetric* if $x_i = x_{a+b-i}$ for all $i \in [a, b]$. See an example of a unimodal sequence in Fig. 2.8.

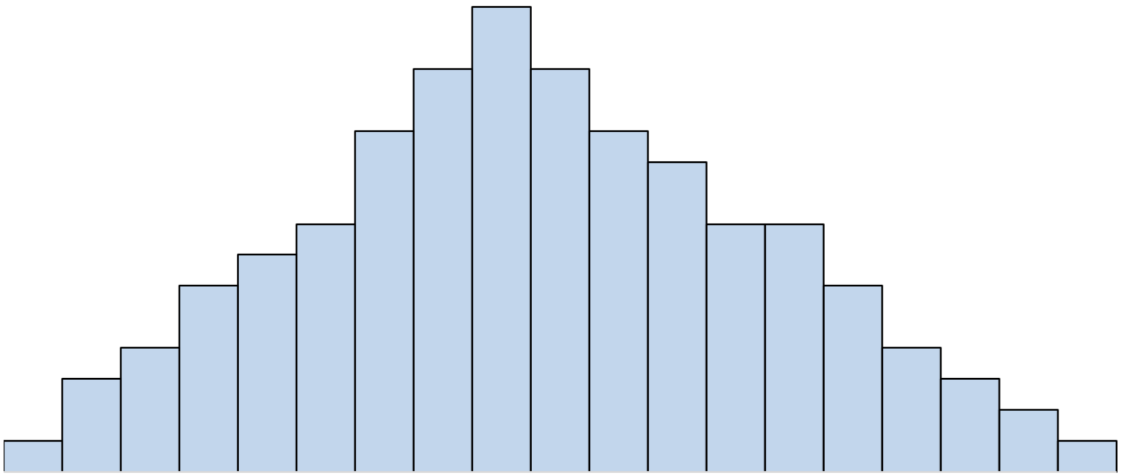

Figure 2.8 – Example of an unimodal sequence represented as a histogram.

Recall $g_d(n, k) := g(n \times k, \ldots, n \times k)$ (repeated d times) for specific rectangular Kronecker coefficients, where $n \times k := (k, \ldots, k)$ (n times). These coefficients were studied in [59]; in particular, for fixed k and odd $d \geqslant 3$ the sequence $\{g_d(n, k)\}_{n=0,\ldots,k^{d-1}}$ is symmetric (note that $g_d(n, k) = 0$ for $n > k^{d-1}$), and the authors made the following conjecture on its unimodality.

Conjecture 2.6.1. *Let $d \geqslant 3$ be odd and k be even. Then the sequence $\{g_d(n, k)\}_{n=0,\ldots,k^{d-1}}$ is unimodal.*

For example, for the following sequences:

$$\{g_3(n, 4)\}_{n=0,\ldots,16} = 1, 1, 1, 2, 5, 6, 13, 14, 18, 14, 13, 6, 5, 2, 1, 1, 1,$$

$$\{g_5(n, 2)\}_{n=0,\ldots,16} = 1, 1, 5, 11, 35, 52, 112, 130, 166, 130, 112, 52, 35, 11, 5, 1, 1.$$

In this section, refinement of this conjecture is provided and proved for $k = 2$.

For a partition $\lambda = (\lambda_1, \ldots)$ and $k \geqslant \lambda_1$ define the operation

$$\rho_k : \lambda \mapsto (k, \lambda) = (k, \lambda_1, \ldots),$$

which adds the first part k to λ (similarly, the inverse operation ρ_k^{-1} removes the first part k). This operation is also applied on tuples of partitions componentwise, i.e. $\rho_k(\lambda^{(1)}, \lambda^{(2)}, \ldots) = (\rho_k\lambda^{(1)}, \rho_k\lambda^{(2)}, \ldots)$.

The following more general conjecture on unimodality of Kronecker coefficients is proposed.

Conjecture 2.6.2. *Let $d \geqslant 3$ be odd, k be even and $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(d)})$ be a d-tuple of partitions of $m \leqslant k^d/2$ each with parts at most k. Let $a = \max_i(\lambda^{(i)})'_1$ and $b = \min_i(\lambda^{(i)})'_k$. Then the sequence*

$$\{g(\rho_k^n \boldsymbol{\lambda})\}_{n=-b,\dots,k^{d-1}-a} \tag{2.22}$$

is unimodal.

Note that $g(\rho_k^n \boldsymbol{\lambda}) = 0$ for $n \notin [-b, k^{d-1} - a]$ by trivial reasons. Moreover, it is shown (see Lemma 2.6.12) that the sequence (2.22) is symmetric if m is divisible by $k/2$ and each $\lambda = \lambda^{(i)}$ satisfies the complementarity condition $\rho_k^{k^{d-1}-2m/k} \lambda = k^{d-1} \times k - \lambda = (k - \lambda_{k^{d-1}}, \dots, k - \lambda_1)$, such partitions (and their tuples) are called *k-complementary.*

Remark 2.6.3. This statement essentially splits the set of d-tuples of partitions with parts at most k into disjoint sequences, so that the corresponding Kronecker coefficients form unimodal and sometimes symmetric sequences.

Remark 2.6.4. This conjecture was verified computationally on many examples, including all sequences with $d = 3, k = 4$ and $m \leqslant 8$. Some examples are given of such unimodal sequences in Example 2.6.28. Computations show that similar unimodality of Kronecker coefficients frequently hold in the case of odd k as well, although there is an example when it does *not* hold: $\{g_3(n, 3)\}_{n=0,1,2,\dots} = \{1, 0, 1, \dots\}$. This is the only case ruining unimodality for odd k that the authors are aware of (from our computations).

Remark 2.6.5. The unimodality can be compared as a specific 'vertical' counterpart of the well-known semigroup property for Kronecker coefficients [39].

It shall be proven Conjecture 2.6.2 (and hence Conjecture 2.6.1) for $k = 2$.

Theorem 2.6.6. *Conjecture 2.6.2 holds for $k = 2$.*

Note that all two-column partitions are 2-complementary. In particular, taking $\lambda^{(i)} = \varnothing$ have $\rho_2^n \varnothing = n \times 2 = (2^n)$ and the theorem says that the sequence $\{g_d(n, 2)\}_{n=0,\dots,2^{d-1}}$ is unimodal (and symmetric).

The Theorem 2.6.6 is proved by showing the *hard Lefschetz property* on highest weight spaces corresponding to Kronecker coefficients.

2.6.1 Lefschetz properties (LP)

Let $V = (\mathbb{C}^k)^{\otimes d}$ which is a k^d-dimensional space of tensors. It is known (see [41, 56]) that for a d-tuple $\boldsymbol{\lambda}$ of partitions of m with parts at most k holds

$$g(\boldsymbol{\lambda}) = \dim \mathrm{HWV}_{\boldsymbol{\lambda}'} \bigwedge^m V$$

is the dimension of the highest weight vector space of weight $\boldsymbol{\lambda}'$ w.r.t. the induced action of the group $\mathrm{GL}(k)^{\times d}$.

Define the element $\omega = \omega_{d,k} \in \bigwedge^k V$ which is called the *Cayley form*[2] as follows:

$$\omega := \sum_{\pi_1,\ldots,\pi_d \in S_k} \mathrm{sgn}(\pi_1 \cdots \pi_d) \bigwedge_{i=1}^{k} e_{\pi_1(i)} \otimes \ldots \otimes e_{\pi_d(i)},$$

which is a unique highest weight vector of weight $(1 \times k)^d$ (moreover, it is also $\mathrm{SL}(k)^{\times d}$-invariant) [59].

It is natural to approach Conjecture 2.6.2 by establishing a corresponding Lefschetz property. Consider the following Lefschetz properties depending on $\boldsymbol{\lambda}$ and k. (The notation here as in Conjecture 2.6.2.)

($\mathrm{LP}_{\boldsymbol{\lambda},k}$) *For some* $n_0 \in [-b, k^{d-1} - a]$ *the map*

$$L : \mathrm{HWV}_{(\rho_k^n \boldsymbol{\lambda})'} \bigwedge V \longrightarrow \mathrm{HWV}_{(\rho_k^{n+1} \boldsymbol{\lambda})'} \bigwedge V, \qquad L : v \longmapsto \omega \wedge v,$$

is injective for each $n \in [-b, n_0)$ *and is surjective for each* $n \in [n_0, k^{d-1} - a)$.

($\mathrm{HLP}_{\boldsymbol{\lambda},k}$) *For each* $n \in [-b, (k^{d-1} - a + b)/2]$ *the map*

$$L^{k^{d-1}-a-2n} : \mathrm{HWV}_{(\rho_k^n \boldsymbol{\lambda})'} \bigwedge V \longrightarrow \mathrm{HWV}_{(\rho_k^{k^{d-1}-2m/k-2n} \boldsymbol{\lambda})'} \bigwedge V, \qquad L : v \longmapsto \omega \wedge v,$$

is an isomorphism.

Let us note that the hard Lefschetz property $\mathrm{HLP}_{\boldsymbol{\lambda},k}$ only makes sense for k-complementary $\boldsymbol{\lambda}$. The interest is for which d, k and $\boldsymbol{\lambda}$ the properties $\mathrm{LP}_{\boldsymbol{\lambda},k}$ and $\mathrm{HLP}_{\boldsymbol{\lambda},k}$ hold. It is clear that $\mathrm{HLP}_{\boldsymbol{\lambda},k}$ implies $\mathrm{LP}_{\boldsymbol{\lambda},k}$. Figure 2.9 represents the scheme of the map L and the sequences it is produces.

In §2.6.4 it is proved that the properties $\mathrm{LP}_{\boldsymbol{\lambda},2}$ and $\mathrm{HLP}_{\boldsymbol{\lambda},2}$ hold for all odd $d \geqslant 3$ and corresponding $\boldsymbol{\lambda}$ (which are all 2-complementary). It is conjectured that the properties $\mathrm{LP}_{\boldsymbol{\lambda},k}$ and $\mathrm{HLP}_{\boldsymbol{\lambda},k}$ hold for all odd $d \geqslant 3$ and even k. It is shown that these properties have two important implications.

a) LP implies Kronecker unimodality

Proposition 2.6.7. *If* $\mathrm{LP}_{\boldsymbol{\lambda},k}$ *holds then Conjecture 2.6.2 holds for* $\boldsymbol{\lambda}$.

b) HLP implies d-dimensional Alon–Tarsi conjecture

Let us now explain the second implication of the Lefschetz property. There are *d-dimensional Alon–Tarsi numbers* $\mathrm{AT}_d(k) \in \mathbb{Z}$ introduced by Bürgisser and Ikenmeyer [41] (for $d = 3$) and studied in [59] (for general d), which are equal to the difference of the number of 'positive' and 'negative' *Latin hypercubes*, see §b) for definitions. Bürgisser and Ikenmeyer [41] showed that $\mathrm{AT}_3(k)$ is an

[2] As shown in [59], ω can be viewed as dual to *Cayley's first hyperdeterminant* [16, 17]

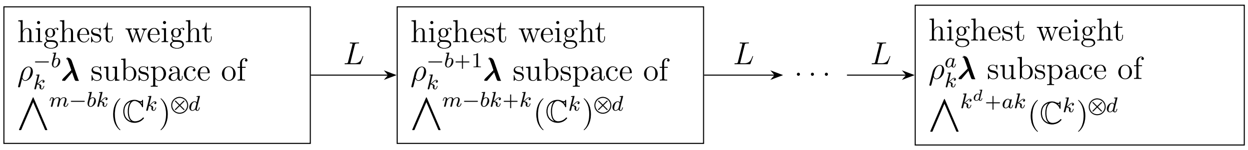

Figure 2.9 – Diagram of subspace sequences corresponding to unimodal sequence of Kronecker coefficients.

evaluation of certain fundamental invariant at unit tensor and they raised the problem whether $\mathrm{AT}_3(k) \neq 0$ holds for even k (having positive computational evidence for $k = 2, 4$); see also [51] for related work. The celebrated Alon–Tarsi conjecture on Latin squares [43] states that $\mathrm{AT}_2(k) \neq 0$ for even k (see also [52–54]). The statement $\mathrm{AT}_d(k) \neq 0$ for even k can be viewed as a d-dimensional Alon–Tarsi conjecture. It is shown that the above Lefschetz property implies this conjecture.

Proposition 2.6.8. *Let $d \geqslant 3$ be odd and k be even. If $\mathrm{HLP}_{\varnothing,k}$ holds then $\mathrm{AT}_d(k) \neq 0$.*

c) More general LP does not hold

To indicate significance of the highest weight spaces for the defined Lefschetz properties, let us consider the following more general Lefschetz property $\mathrm{LP}_{d,k}$ depending on d, k.

($\mathrm{LP}_{d,k}$) *For each $n \in [0, k^d - k]$ the map*

$$L : \bigwedge^{n} V \longrightarrow \bigwedge^{n+k} V, \qquad L : v \longmapsto \omega \wedge v,$$

has full rank.

While it is proved that the property $\mathrm{LP}_{d,k}$ holds for $k = 2$, is is also shown that for $k > 2$ it does *not* hold, see §2.6.5 for details.

Theorem 2.6.9. *Let $d \geqslant 3$ be odd and $k > 2$. Then the property $\mathrm{LP}_{d,k}$ does not hold.*

2.6.2 Highest weight algebra

It is shown that for $V = (\mathbb{C}^k)^{\otimes d}$ the highest weight space $\mathrm{HWV} \bigwedge V = \bigoplus_{\boldsymbol{\lambda}} \mathrm{HWV}_{\boldsymbol{\lambda}} \bigwedge V$ can be viewed as a subalgebra[3] of $\bigwedge V$, where $\boldsymbol{\lambda}$ runs over d-tuples of partitions of m. This fact is an important ingredient for using the Lefschetz property on the general spaces to obtain Kronecker unimodality in light of (2.19).

The group $G = \mathrm{GL}(k)^{\times d}$ has the dual representation V^* defined by the action $g \cdot v^* = v^*(g^{-1}x)$ for $g \in G, v \in V$. If v is the highest weight vector of weight λ then v^* is the highest weight vector of weight $-w_0\lambda$ where w_0 is the

[3]Highest weight vector by definition is nonzero, but 0 is formally added to make this set subalgebra.

longest element of the corresponding Weyl group. In our case if $v \in \mathrm{HWV}_{\boldsymbol{\lambda}} \bigwedge V$ then $v^* \in \mathrm{HWV}_{\boldsymbol{\lambda}^*} \bigwedge V^*$, where for a partition $\lambda^* = (-\lambda_k, \ldots, -\lambda_1)$ and for a tuple $\boldsymbol{\lambda}^* = ((\lambda^{(1)})^*, \ldots, (\lambda^{(d)})^*)$. Also write $\lambda - \mu := \lambda + \mu^*$. All above extends to tuples of partitions or sequences coordinatewise.

Lemma 2.6.10. *Let* $v \in \mathrm{HWV}_{\boldsymbol{\lambda}} \bigwedge^n V$ *and* $u \in \mathrm{HWV}_{\boldsymbol{\mu}} \bigwedge^m V$*. Then*
1 (exterior product) $u \wedge v \in \mathrm{HWV}_{\boldsymbol{\lambda}+\boldsymbol{\mu}} \bigwedge^{n+m} V$*.*
2 (interior product) if $\boldsymbol{\lambda} - \boldsymbol{\mu} \geqslant 0$ *then* $u^*(v) \in \mathrm{HWV}_{\boldsymbol{\lambda}-\boldsymbol{\mu}} \bigwedge^{n-m} V$*.*

Proof. Let $E \in U(k)^{\times d}$ be unipotent operator. Then $E(v \wedge u) = E(v) \wedge E(u) = v \wedge u$, i.e. $v \wedge u$ is a highest weight vector of weight $\boldsymbol{\lambda}+\boldsymbol{\mu}$. Similarly, $E(u^* \wedge v) = E(u^*) \wedge E(v) = u^* \wedge v$. The latter relies on the fact that $a^*(b) = a^*(E^{-1}Eb) = (Ea^*)(Eb)$. □

Remark 2.6.11. One of the examples of highest weight vectors were considered in [56] and called 'pyramids'. Objects appeared there also are referred to as higher dimensional partitions, which were studied in case $d = 2$ in [61] and for general d in [62].

a) Hodge duality

It is known that the space $\bigwedge V$ has natural *Hodge duality.* The *Hodge star operator* is defined as contraction with volume form:

$$\star : \bigwedge^{n} V \to \bigwedge^{k^d-n} V, \quad v \mapsto v^* \wedge \mathrm{vol}$$

for any $n \in [k^d]$, where $\mathrm{vol} := e_{[k]^d}$ is the volume form, which also a highest weight vector. It is known that $\star \circ \star = (-1)^{n(k^d-n)}\mathrm{id}$ is a scalar multiple of an identity operator.

Lemma 2.6.12. *Let* $\boldsymbol{\lambda}$ *be a d-tuple of partitions of* m *with parts at most* k *and lengths at most* k^{d-1}*. Then one has the following symmetry of Kronecker coefficients:*

$$g(\boldsymbol{\lambda}) = g((k^{d-1} \times k)^d - \boldsymbol{\lambda}).$$

Proof. By Lemma 2.6.10, the Hodge star operator sends the highest weight vector to the highest weight vector (if non-zero). But since $\star \circ \star = \pm\mathrm{id}$, the operator $\star$ defines an isomorphism between the highest weight spaces $\star : \mathrm{HWV}_{\boldsymbol{\lambda}'} \bigwedge V \to \mathrm{HWV}_{(k \times k^{d-1})^d - \boldsymbol{\lambda}'} \bigwedge V$. □

Remark 2.6.13. For $d = 3$ this symmetry of Kronecker coefficients was shown in [37, Cor. 4.4.15] (by a different argument). Here it is shown it for all odd d and provide a direct proof. This lemma can also be generalized if replace $V = (\mathbb{C}^k)^{\otimes d}$ to more general space $\mathbb{C}^{k_1} \otimes \ldots \otimes \mathbb{C}^{k_d}$.

2.6.3 Proofs of Propositions 2.6.7 and 2.6.8

It will now be provided the proofs of Proposition 2.6.7 and Proposition 2.6.8.

Proof of Proposition 2.6.7. Without loss of generality, one can shift the sequence and assume $b = 0$ (if $b > 0$ then replace $\boldsymbol{\lambda} \to \rho_k^{-b}\boldsymbol{\lambda}$), since the statement $\mathrm{LP}_{\boldsymbol{\lambda},k}$ is common for each element of the sequence $\{\rho_k^n\boldsymbol{\lambda}\}_{n\in[-b,k^{d-1}-a]}$.

Let us denote the spaces

$$U_n := \mathrm{HWV}_{\boldsymbol{\lambda}'+(k\times n)^d} \bigwedge^{m+nk} V, \qquad n = 0, \dots, k^{d-1} - a,$$

so that

$$\dim U_n = g((\boldsymbol{\lambda}' + (k \times n)^d)') = g(\rho_k^n\boldsymbol{\lambda}).$$

By Lemma 2.6.10 one has

$$L : U_n \to U_{n+1}, \quad v \mapsto \omega \wedge v.$$

By $\mathrm{LP}_{\boldsymbol{\lambda},k}$ the sequence of maps

$$U_0 \xrightarrow{L} U_1 \xrightarrow{L} \dots \xrightarrow{L} U_{k^{d-1}-a}$$

is injective up to some $n_0 \in [0, k^{d-1} - a]$ and surjective afterwards. This establishes unimodality of the sequence $\{g(\rho_k^n\boldsymbol{\lambda})\}_{n=0,\dots,k^{d-1}-a}$. □

Remark 2.6.14. Each d-tuple $\boldsymbol{\lambda}$ of partitions of m that fit in the rectangle $k^{d-1}\times k$ produces an element of certain unimodal sequence of corresponding Kronecker coefficients, and these sequences do not intersect. Indeed, let λ be a partition in $\boldsymbol{\lambda}$ with the largest $\ell(\lambda) \to \max$. Then one construct the corresponding Kronecker sequence starting from $g(\boldsymbol{\lambda})$ and appending to the left d-tuples of partitions resulted by removing the (largest) part k (until possible); and appending to the right of $g(\boldsymbol{\lambda})$ partitions resulting by inserting part k (until possible).

Proof of Proposition 2.6.8. Using the hard Lefschetz property $\mathrm{HLP}_{\varnothing,k}$ for $n = k^{d-1}/2$ and the result in Theorem 2.5.24(iii) one obtain

$$L^{k^{d-1}}(1) = \omega^{\wedge k^{d-1}} = \pm\mathrm{AT}_d(k) \cdot \mathrm{vol} \neq 0,$$

and hence $\mathrm{AT}_d(k) \neq 0$. □

2.6.4 Proof of Theorem 2.6.6 via $\mathfrak{sl}(2)$ representation

In this section one specialize $k = 2$ so that $V = (\mathbb{C}^2)^{\otimes d}$ for odd $d \geqslant 3$. It will be first proved that $\bigwedge V$ is an $\mathfrak{sl}(2)$ representation.

Recall that the Lie algebra $\mathfrak{sl}(2)$ is given by its basis X, Y, H called *raising, lowering and counting operators* subject to commutation relations

$$[X, Y] = H, \quad [H, Y] = -2Y, \quad [H, X] = 2X. \tag{2.23}$$

Thus, it is enough to show that $\mathrm{End}(\bigwedge V)$ contains a Lie subalgebra isomorphic to $\mathfrak{sl}(2)$. In fact, it will be shown it w.r.t. the carefully chosen operators X, Y that in turn have the highest weight algebra as an invariant subspace.

The key is to present ω in a suitable way.

Definition 2.6.15. Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{C}^2$. Then $V = (\mathbb{C}^2)^{\otimes d} = \langle e_{\mathbf{i}} : \mathbf{i} \in [2]^d \rangle$ where $e_{\mathbf{i}} = e_{i_1} \otimes \ldots \otimes e_{i_d}$ for $\mathbf{i} = (i_1, \ldots, i_d)$. For $\mathbf{i} = (i_1, \ldots, i_d) \in [2]^d$ denote $|\mathbf{i}| := i_1 + \ldots + i_d - d$. Let us separate the *'positive'* coordinates $I^+ := \{\mathbf{i} \in [2]^d : |\mathbf{i}| \text{ is even}\}$ and the *'negative'* $I^- := [2]^d \backslash I^+$. For $\mathbf{i} \in I^+$ denote $\bar{\mathbf{i}} := (3 - i_1, \ldots, 3 - i_d) \in I^-$. For instance, $(1, \ldots, 1) \in I^+$ and its negation $(2, \ldots, 2) \in I^-$. It is clear that this 'negation' of indices is a bijection, i.e. $\overline{I^+} = I^-, \overline{I^-} = I^+$.

Lemma 2.6.16. *For $k = 2$ the Cayley vector $\omega = \omega_{d,2}$ and its dual ω^* can be written as follows:*

$$\omega = \sum_{\mathbf{i} \in I^+} e_{\mathbf{i}} \wedge e_{\bar{\mathbf{i}}}, \qquad \omega^* = \sum_{\mathbf{i} \in I^+} e^*_{\mathbf{i}} \wedge e^*_{\bar{\mathbf{i}}}.$$

Proof. By the definition of $\omega = \omega_{d,2}$ one has

$$\omega = \sum_{\pi = (\pi_1, \ldots, \pi_d) \in (S_2)^d} \mathrm{sgn}(\pi_1) \cdots \mathrm{sgn}(\pi_d)\, e_{\pi(1)} \wedge e_{\pi(2)}.$$

But note that $\pi(1) = (\pi_1(1), \ldots, \pi_d(1)) = (3 - \pi(2), \ldots, 3 - \pi(2)) = \overline{\pi(2)}$ and the sign $\mathrm{sgn}(\pi_1) \cdots \mathrm{sgn}(\pi_d) = (-1)^{|\pi(1)|}$. Hence

$$\omega = \sum_{(1,\mathbf{i}) \in I^+} e_{(1,\mathbf{i})} \wedge e_{(2,\bar{\mathbf{i}})} - \sum_{(2,\mathbf{i}) \in I^+} e_{(1,\bar{\mathbf{i}})} \wedge e_{(2,\mathbf{i})} = \sum_{\mathbf{i} \in I^+} e_{\mathbf{i}} \wedge e_{\bar{\mathbf{i}}}$$

and similarly for ω^*. □

Now using these elements define the raising operator $X = L : \bigwedge V \to \bigwedge V$ by the left exterior product with ω, the lowering operator $Y : \bigwedge V \to \bigwedge V$ by the left interior product with ω^*, i.e.

$$X : v \mapsto \omega \wedge v, \qquad Y : v \mapsto \omega^* \wedge v,$$

and the counting operator H that reduces to multiplication by scalar $(\ell - 2^{d-1})$ on $\bigwedge^\ell V$.

Lemma 2.6.17. *The operators X, Y, H satisfy the commutation relations (2.23).*

Proof. Since the element ω is now written in a convenient way, the same proof as can be found in [63, Ch. VIII, p. 207] works here, which is presented for completeness.

Choose the total lexicographic order $<$ on the set of our indices $[2]^d$. Let us select the following basis for the space $\bigwedge V$:

$$e_{A,B,C} = e_{a_1} \wedge \ldots \wedge e_{a_p} \wedge e_{\overline{b_1}} \wedge \ldots \wedge e_{\overline{b_q}} \wedge e_{c_1} \wedge e_{\overline{c_1}} \wedge \ldots \wedge e_{c_m} \wedge e_{\overline{c_m}}$$

where $A = \{a_1 < \ldots < a_p\}, B = \{b_1 < \ldots < b_q\}, C = \{c_1 < \ldots < c_m\}$ are disjoint subsets of I^+. Using the fact that $v \wedge e_{\mathbf{i}} \wedge e_{\bar{\mathbf{i}}} = e_{\mathbf{i}} \wedge e_{\bar{\mathbf{i}}} \wedge v$ it is easy to verify that

$$Xe_{A,B,C} = \sum_{\mathbf{j} \in I^+ \setminus A \cup B \cup C} e_{A,B,C \cup \{\mathbf{j}\}},$$
$$Ye_{A,B,C} = \sum_{\mathbf{j} \in C} e_{A,B,C \setminus \{\mathbf{j}\}}.$$

This allows us to calculate commutators. Assume $e_{A,B,C} \in \bigwedge^\ell V$, i.e. $p + q + 2m = \ell$; then one has

$$[X,Y](e_{A,B,C}) = \sum_{\mathbf{j}_1 \in C} \sum_{\mathbf{j}_2 \in I^+ \setminus A \cup B \cup (C \setminus \{\mathbf{j}_1\})} e_{A,B,(C \setminus \{\mathbf{j}_1\}) \cup \{\mathbf{j}_2\}}$$
$$- \sum_{\mathbf{j}_2 \in I^+ \setminus A \cup B \cup C} \sum_{\mathbf{j}_1 \in (C \cup \{\mathbf{j}_2\})} e_{A,B,(C \cup \{\mathbf{j}_2\}) \setminus \{\mathbf{j}_1\}}.$$

If $\mathbf{j}_1$ and $\mathbf{j}_2$ differ then the coefficient at $\tilde{e}_{A,B,(C \cup \{\mathbf{j}_2\}) \setminus \{\mathbf{j}_1\}}$ is 0, since the order of adding $\mathbf{j}_2$ and removing $\mathbf{j}_1$ does not matter. If $\mathbf{j}_1 = \mathbf{j}_2 = \mathbf{j}$, the coefficient coming from the first sum is m and from the second sum is $2^{d-1} - p - q - m$, thus

$$[X,Y](e_{A,B,C}) = (\ell - 2^{d-1}) e_{A,B,C} = He_{A,B,C}.$$

The remaining two relations are straightforward. □

Definition 2.6.18. Define $P^n = \ker(Y) \cap \bigwedge^n V$ called the *primitive class* of degree n, where it is set $P^n = 0$ for $n < 0$. The elements of P^n are called *primitive vectors.*

The following are standard known properties of $\mathfrak{sl}(2)$ representation.

Proposition 2.6.19. *One has:*

1 $[X^i, Y](v) = i(n - 2^{d-1} + i - 1)X^{i-1}(v)$ *for* $v \in \bigwedge^n V$.

2 *For* $n \in [0, 2^d]$

$$\bigwedge^n V = \bigoplus_{i=0}^{\lfloor n/2 \rfloor} X^i P^{n-2i}, \quad \bigwedge V = \bigoplus_{n \geqslant 0} \bigoplus_{i=0}^{2^{d-2}-n} X^i(P_n)$$

called Lefschetz decomposition.

3 $P^n = 0$ *for* $n > 2^{d-1}$.

Proof. (i) For $i = 1$, the statement is proved earlier. Now by induction on i one gets

$$\begin{aligned}
[X^i, Y](v) &= (X^iY - YX^i)(v) \\
&= (X^iY - XYX^{i-1} + XYX^{i-1} - YX^i)(v) \\
&= X[X^{i-1}, Y]v + [X, Y]X^{i-1}(v) \\
&= ((i-1)(n - 2^{d-1} + i - 2) + n + 2(i-1) - 2^{d-1})X^{i-1}(v) \\
&= i(n - 2^{d-1} + i - 1)X^{i-1}(v).
\end{aligned}$$

(ii) Since $\bigwedge V$ is a finite-dimensional $\mathfrak{sl}(2)$ representation, it is a direct sum of irreducible $\mathfrak{sl}(2)$ representations. Any irreducible representation $W \subseteq \bigwedge V$ contains a primitive vector $v \in P^n$ of degree n, i.e. $Yv = 0$ (if $Yv \neq 0$ then replace v with $Y^\ell v$ for large enough ℓ). Let ℓ be minimal integer such that $X^{\ell+1}v = 0$. Then by (i) the space $\langle v, Xv, \ldots, X^\ell v\rangle$ is stable under the action of X,Y and H, hence is irreducible $\mathfrak{sl}(2)$ representation and any irreducible is of that form. Moreover, by (i) with $i \to \ell + 1$ one gets $0 = [X^{\ell+1}, Y](v) = (\ell+1)(n-2^{d-1}+\ell)X^\ell(v)$, hence $\ell = 2^{d-1} - n$ and the irreducible representation W is stretched from $\bigwedge^n V$ up to $\bigwedge^{2^d-n} V$. This implies the decomposition.

(iii) Let $v \in P^n$ and i be minimal with $X^iv = 0$. For $n > 2^{d-2}$, by (i) one has $[X^i, Y](v) = i(2n - 2^{d-1} + i - 1)X^{i-1}(v)$, which implies $i = 0$, i.e. $v = 0$. □

An immediate implication of the Lefschetz decomposition is the following.

Corollary 2.6.20. *Let $d \geqslant 3$ be odd. Then the property* $\mathrm{LP}_{d,2}$ *holds.*

a) Highest weight subspace

Let $\boldsymbol{\lambda}$ be a d-tuple of partitions with at most two columns; for simplicity let us also assume that $(\lambda^{(1)})' = (m)$ for $m \leqslant 2^{d-1}$ (see Remark 2.6.14). Note that one has

$$g(\rho_2^n \boldsymbol{\lambda}) = \dim \mathrm{HWV}_{\boldsymbol{\lambda}'+(2\times n)^d} \bigwedge^{m+2n} V.$$

Let us denote the spaces

$$U_{m+2n} := \mathrm{HWV}_{\boldsymbol{\lambda}'+(2\times n)^d} \bigwedge^{m+2n} V, \qquad U := \bigoplus_{i=0}^{2^{d-1}-m} U_{m+2n}$$

for $n = 0, \ldots, 2^{d-1} - m$.

Recall that the vector ω is a unique highest weight vector of weight $(2 \times 1)^d$ in the space $\bigwedge^2 V$.

Lemma 2.6.21. *The space U is an $\mathfrak{sl}(2)$ representation.*

Proof. Since X and Y multiply and contract with the highest weight vector $\omega = \omega_{d,2}$, by Lemma 2.6.10 one has: $X(U_i) \subseteq U_{i+1}, Y(U_i) \subseteq U_{i-1}$ and $H(U_i) \subseteq$

U_i. Therefore, U is an invariant subspace for the action of $\mathfrak{sl}(2)$, hence is a representation itself. $\square$

The above result implies Theorem 2.6.6 by standard properties of $\mathfrak{sl}(2)$. Let us illustrate how. Denote $Q^{m+2n} := P^{m+2n} \cap U$ as primitive class of representation U. Then (ii) and (iii) of Proposition 2.6.19 does also hold if $\bigwedge V$ and P^ℓ is replaced by U and Q^ℓ.

Corollary 2.6.22 (Hard Lefschetz property for $k = 2$)**.** *The map $X^\ell : U_{2^{d-1}-\ell} \to U_{2^{d-1}+\ell}$ is an isomorphism for $\ell = 0, \dots, m/2$ if m is even and for $\ell = 1, \dots, \lfloor m/2 \rfloor$ if m is odd.*

Proof. By the Lefschetz decomposition in Proposition 2.6.19(ii) one has:

$$\begin{aligned} U_{2^{d-1}+\ell} &= \bigoplus_{i\geqslant 0} X^i Q^{2^{d-1}+\ell-2i} \\ &= X^\ell \left(\bigoplus_{i\geqslant 0} X^i q^{2^{d-1}-\ell-2i} \right) \bigoplus \left(\bigoplus_{i=0}^{\ell-1} X^i Q^{2^{d-1}+\ell-2i} \right) \\ &= X^\ell U_{2^{d-2}-\ell} \bigoplus 0 = X^\ell Q_{2^{d-2}-\ell} \end{aligned}$$

where the terms $X^i Q^{2^{d-1}+\ell-2i}$ vanish: for $i < \ell/2$ since $Q^{2^{d-2}+\ell-i} = 0$ by Proposition 2.6.19(iii), and for $\ell/2 \leqslant i < \ell$ since for $v \in Q^{2^{d-2}+\ell-2i}$ by Proposition 2.6.19(ii) $X^i v = 0$. By Lemma 2.6.12 one has $\dim U_{2^{d-2}-\ell} = \dim U_{2^{d-2}+\ell}$ and the statement follows. $\square$

Proof of Theorem 2.6.6. By Corollary 2.6.22 the map $U_{m+2i} \xrightarrow{X} U_{m+2(i+1)}$ is injective for $n < (2^{d-1} - m)/2$ and surjective for $n \geqslant (2^{d-1} - m)/2$. Note that for a partition $\lambda = \lambda^{(i)}$ with at most two columns k-complementary condition always holds: $2 \times 2^{d-2} - \lambda' = \lambda' + 2 \times (2^{d-2} - m)$. Therefore, by Proposition 2.6.7 the sequence is symmetric and unimodal. $\square$

Remark 2.6.23. One can similarly define the operators X, Y, H for $k = 4$ using $\omega_{d,4}$. However, the commutation relation $[X, Y] = H$ from (2.23) does *not* hold in this case, i.e. these operators do not form an $\mathfrak{sl}(2)$ representation. For instance, if one takes $T = \{(111), (222)\} \subseteq [4]^3$ then

$$[X, Y](e_T) = 148 e_{\{(111),(222)\}} + 4 e_{\{(112),(221)\}} + 4 e_{\{(121),(212)\}} - 4 e_{\{(122),(211)\}}$$

and so the commutator does not rescale this vector.

Remark 2.6.24. Another implication of $\mathfrak{sl}(2)$ structure on $\bigwedge V$ with the element $\omega_{d,2}$ is the following unimodality of magic sets in the cube $[2]^d$. Let $b_d(n) := |B_{d,2}(n)|$ be the number of magic sets of magnitude n in $[2]^d$ (see Section b)). Then the sequence $\{b_d(n)\}_{n=0,\dots,2^{d-1}}$ is unimodal (and symmetric) since the space of weight vectors $\bigoplus_n \mathrm{WV}_{(2\times n)^d} \bigwedge^{2n} V$ is preserved by the $\mathfrak{sl}(2)$ triplet (X, Y, H).

2.6.5 More general LP does not hold

It is now proved that the Lefschetz property $\mathrm{LP}_{d,k}$ (defined in §c)) does not hold for $k > 2$.

Theorem 2.6.25. *Let $d \geqslant 3$ be odd and $k > 2$. Then the property $\mathrm{LP}_{d,k}$ does not hold.*

Proof. By unimodality of the sequence $\{\dim \bigwedge^n V\}_{n\in[0,k^d]} = \{\binom{k^d}{n}\}_{n\in[0,k^d]}$ the map

$$L : \bigwedge^{n} V \to \bigwedge^{n+k} V$$

must be injective for $n \leqslant (k^d - k)/2$ and surjective afterwards. It will be shown that the map L has a non-empty kernel for some $n < (k^d - k)/2$.

Consider the following vector

$$v := \bigwedge_{\mathbf{i}\in[k]^{d-1}} e_{1\mathbf{i}}$$

which is a basis vector corresponding to the set of all entries $\mathbf{i}$ of the cube $[k]^d$ that lie in the first (or any) slice of the first direction. Note that v is a $\mathrm{GL}(k)^{\times d}$ highest weight vector, since it nulls under the action of corresponding raising operators (see e.g. for a similar argument [56]). Then

$$L(v) = \omega \wedge v = \sum_{\pi=(\pi_1,\dots,\pi_d)\in(S_k)^d} \mathrm{sgn}(\pi) \bigwedge_{i=1}^{k} e_{\pi(i)} \wedge v = 0$$

since for some j one has $\pi_1(j) = 1$ and $e_{\pi(j)} \wedge v = 0$. □

Remark 2.6.26. One can construct many other highest weight vectors like v by appending the highest weight vectors to the slices $2, \dots, k$ in the first direction; all of them will null under L by the same reason.

Remark 2.6.27. Let us now make a few remarks on the related property $\mathrm{LP}_{\boldsymbol{\lambda},k}$.

(i) The vector v in the proof above has weight $\boldsymbol{\lambda}' = (1\times k^{d-1}, (k\times k^{d-2})^{d-1})$. Then $g(\boldsymbol{\lambda}) = 1$ and the property $\mathrm{LP}_{\boldsymbol{\lambda},k}$ gives unimodality of the sequence with only element.

(ii) Choose $n = \lfloor \frac{d-1}{d} k \rfloor + 1$. Let us take $\boldsymbol{\lambda} = (n^{d-1} \times n)^d$ and let $u = e_{[n]^d} = \bigwedge_{\mathbf{i}\in[n]^d} e_{\mathbf{i}}$. Then by analogous reasoning $u \in \mathrm{HWV}_{(\boldsymbol{\lambda})'} \bigwedge V$. Note that

$$L(v) = \omega \wedge v = \sum_{\pi=(\pi_1,\dots,\pi_d)\in(S_k)^d} \mathrm{sgn}(\pi) \bigwedge_{i=1}^{k} e_{\pi(i)} \wedge v = 0,$$

since for each $\pi \in (S_k)^d$ there is $i \in [k]$ with $\pi(i) \in [n]^d$; indeed, the d permutations $\pi_1, \dots, \pi_d$ in π have $d(k-n)$ numbers larger than n, and since $d(k-n) < k$

there must be $i \in [k]$ for which $\pi(i) = (\pi_1(i), \ldots, \pi_d(i)) \in [n]^d$. Hence, $L(v) = 0$, but the property $\mathrm{LP}_{\boldsymbol{\lambda},k}$ will still hold if one has the corresponding sequence $\{g(\rho_k^\ell \boldsymbol{\lambda})\}_{\ell \geqslant 0} = \{1, 0, 0, \ldots\}$ (which holds e.g. for $k = 4$ and $n = 3$).

(iii) Finally, for odd k the situation is different. For instance, the property $\mathrm{LP}_{\varnothing,k}$ does not hold, since $L(\omega) = \omega^2 = 0$ by Theorem 2.5.24.

2.6.6 Some examples

Some examples of sequences $\{g(\rho_k^n \boldsymbol{\lambda})\}$ are illustrated.

Example 2.6.28. Let us give some examples showing unimodality of $\{g(\rho_k^n \boldsymbol{\lambda})\}$.

(a) (Unimodal and symmetric) Let $\boldsymbol{\lambda} = (42, 222, 321)$ so that for each partition λ in this tuple one has $16 \times 4 - \lambda = \rho_4^{13}\lambda$, e.g. $16 \times 4 - (42) = (4^{14}2)$. Then the sequence is unimodal and symmetric:

$$\{g(\rho_4^n \boldsymbol{\lambda})\}_{n=0,\ldots,13} = 1, 15, 128, 728, 2684, 6395, 9884, 9884, 6395, 2684, 728, 128, 15, 1.$$

(b) (Unimodal and not symmetric) Let $\boldsymbol{\lambda} = (32, 221, 41)$. Then the sequence is unimodal but not symmetric:

$$\{g(\rho_4^n \boldsymbol{\lambda})\}_{n=0,\ldots,13} = 1, 8, 54, 281, 1027, 2531, 4179, 4584, 3331, 1613, 521, 114, 18, 2.$$

(c) (Unimodality for odd k) While unimodality only for even k is discussed, it is frequently true for odd k as well. For instance, let $\boldsymbol{\lambda} = (32, 221, 311)$. Then the sequence is unimodal:

$$\{g(\rho_3^n \boldsymbol{\lambda})\}_{n=0,\ldots,6} = 1, 4, 7, 7, 5, 3, 1.$$

Although for odd k and $\boldsymbol{\lambda} = (1 \times k)^3$ it is known that $g(\boldsymbol{\lambda}) = 1$ but $g(\rho_k \boldsymbol{\lambda}) = 0$ which shows that the sequence is not always unimodal in this case.

Example 2.6.29 (Log-concavity)**.** Apart from unimodality, such sequences are frequently log-concave as well, that is when $g(\rho_k^n \boldsymbol{\lambda})^2 \geqslant g(\rho_k^{n-1} \boldsymbol{\lambda}) \cdot g(\rho_k^{n+1} \boldsymbol{\lambda})$. For instance, the sequences from Example 2.6.28 are all log-concave. But this is not always true. For example, let $\boldsymbol{\lambda} = ((1), (1), (1))$, then the following sequence is not log-concave but becomes so eventually:

$$\{g(\rho_4^n \boldsymbol{\lambda})\}_{n=0,\ldots,15} = 1, 1, 2, 6, 19, 58, 120, 179, 195, 145, 77, 30, 9, 2, 1, 1,$$

where the log-concavity inequality holds for $n \in [4, 12]$. It is interesting to find for which $\boldsymbol{\lambda}$ the corresponding sequences of Kronecker coefficients are log-concave.

Another curious example is an 'alternating' log-concavity observed in the following rectangular case:

$$\{g_5(n, 2) = g(\rho_2^n \varnothing)\}_{n=0,\ldots,16}$$

$$= 1, 1, 5, 11, 35, 52, 112, 130, 166, 130, 112, 52, 35, 11, 5, 1, 1,$$

where the log-concavity inequality holds if n is even and the reversed (log-convexity) inequality holds if n is odd. Similarly, there is an example of an eventual 'alternating' log-concavity in the following sequence:

$$\{g_3(n, 4) = g(\rho_4^n \varnothing)\}_{n=0,\ldots,16} = 1, 1, 1, 2, 5, 6, 13, 14, 18, 14, 13, 6, 5, 2, 1, 1, 1,$$

where a similar alternating pattern holds for $n \in [4, 12]$. All sequences in our computations were observed to have such types of log-concavity properties.

2.6.7 Conclusion

The study presented in this section establishes several key results regarding the unimodality of Kronecker coefficients. The conjecture on the unimodality of rectangular Kronecker coefficients $g_d(n, k)$ for the case $k = 2$ has been refined and proved. The fundamental result is Theorem 2.6.6, which demonstrates that the sequence $\{g_d(n, 2)\}_{n=0,\ldots,2^{d-1}}$ is unimodal (and symmetric), utilizing the hard Lefschetz property on highest weight spaces corresponding to Kronecker coefficients.

Additionally, a more general conjecture, Conjecture 2.6.2, extends the unimodality property to a broader class of Kronecker coefficients $\{g(\rho_k^n \boldsymbol{\lambda})\}_{n=-b,\ldots,k^{d-1}-a}$. Computational verification of this conjecture has been provided for several specific cases, including all sequences with $d = 3$, $k = 4$, and $m \leqslant 8$.

The Lefschetz properties $\mathrm{LP}_{\boldsymbol{\lambda},k}$ and $\mathrm{HLP}_{\boldsymbol{\lambda},k}$ were introduced, establishing their implications for the unimodality of Kronecker coefficients and the d-dimensional Alon-Tarsi conjecture. Proposition 2.6.7 shows that $\mathrm{LP}_{\boldsymbol{\lambda},k}$ implies Conjecture 2.6.2, while Proposition 2.6.8 demonstrates that $\mathrm{HLP}_{\varnothing,k}$ implies $\mathrm{AT}_d(k) \neq 0$ for even k.

Furthermore, the structure of the highest weight algebra $\mathrm{HWV} \bigwedge V$ has been explored, showing that it can be viewed as a subalgebra of $\bigwedge V$, and the significance of the Hodge duality in establishing the symmetry of Kronecker coefficients has been discussed.

While the general Lefschetz property $\mathrm{LP}_{d,k}$ does not hold for $k > 2$, the specific property $\mathrm{LP}_{\boldsymbol{\lambda},k}$ can still hold for even k and various $\boldsymbol{\lambda}$, suggesting that the unimodality of Kronecker coefficients is a more nuanced phenomenon.

Overall, these results provide a deeper understanding of the algebraic and combinatorial structures underlying Kronecker coefficients, paving the way for future research into their broader implications and applications in computer science, mathematics and physics.

3 APPLICATIONS TO QUANTUM INFORMATION THEORY

In quantum information theory, understanding entangled states complexity within the context of SLOCC (stochastic local operations and classical communications) with d qubits (or qunits) relies on classifying them via local symmetry groups. Resulting classes may be separated by means of invariant polynomials and the values at these polynomials may be used as a measure of entanglement. This paper proposes a method of obtaining invariant polynomials of smallest degrees, that allows one to efficiently characterize SLOCC classes of entangled quantum states. As an application the derivation of minimal degree invariants in special cases is provided.

Understanding entanglement is a basic idea in quantum information theory. It's seen as an important part of quantum information and become an important field of research [64–66]. The main issue is figuring out how to measure and sort the entanglement in quantum states [67]. Polynomial functions that stay the same with stochastic local operations and classical communication (SLOCC) changes have been investigated extensively over the past years [64, 68, 69]. These functions are sometimes exploited to measure the entanglement [34]. Stochastic Local Operations and Classical Communication (SLOCC) is a pivotal concept in this context, offering a framework for classifying entangled states based on their convertibility through local operations and classical communication. This classification is vital because it helps identify which quantum states can be transformed into each other using local operations, shedding light on the fundamental structure of quantum entanglement and its implications for quantum information processing. Within the SLOCC framework, the complexity of entangled states, particularly with systems composed of d quantum units (qunits, n states), becomes a critical area of study. The challenge lies in efficiently categorizing these states to understand their potential for various quantum information tasks. This paper addresses the challenge of classification of entangled states under SLOCC, for odd number $d \geqslant 3$ of parties each having a single qunit. It introduces an improved method for the derivation of invariant polynomials of smallest degrees, which serve as a robust tool for efficiently characterizing SLOCC classes of entangled quantum states [16–18, 64, 68]. This way of derivation was shown in [41, 44] and developed in [49]. Using representation theory, in particular Schur-Weyl duality, the spanning set of homogeneous invariant polynomials of fixed degree over state space of d qunits is obtained.

The main problem addressed in this paper can be stated as follows.

Problem (Orbit separation problem)**.** For two entangled states is it possible to transform one to another by stochastic local operations?

The problem can be reformulated it in the mathematical setting. Quantum states of d particles of n states (d qunits) are interpreted mathematically as elements of $V = \mathbb{C}^n \otimes \ldots \otimes \mathbb{C}^n$ (repeated d times) scaled to unit norm. Under the fixed basis, each state can be associated with d-dimenisonal hypermatrix $\{A_{i_1 \ldots i_d}\}$. Stochastic local operations are associated with the elements of the (SLOCC) group $G = \mathrm{SL}(n) \times \ldots \times \mathrm{SL}(n)$ (repeated d times), where each group copy acts independently on the corresponding tensor component by left multiplication. Here $\mathrm{SL}(n)$ is the group of $n \times n$ matrices of a determinant 1. Thus, SLOCC classes are exactly the orbits of a group action.

Invariant polynomials discussed in Section 1.2 are then a useful tool for distinguishing the orbits of tensors. But there are two problems one may face:

1 evaluation of invariant polynomials may be a computationally expensive, since the polynomials are hard to compute;

2 the search space (of invariant polynomials) may be very large.

Most of the problems related to Tensors are NP-hard as shown in [70], see Table 3.1.

Table 3.1 – Complexity of problems related to 3-tensors [70].

Problem	**Complexity**
Bivariate Matrix Functions over $\mathbb{R}, \mathbb{C}$	Undecidable
Bilinear System over $\mathbb{R}, \mathbb{C}$	NP-hard
Eigenvalue over $\mathbb{R}$	NP-hard
Approximating Eigenvector over $\mathbb{R}$	NP-hard
Symmetric Eigenvalue over $\mathbb{R}$	NP-hard
Approximating Symmetric Eigenvalue over $\mathbb{R}$	NP-hard
Singular Value over $\mathbb{R}, \mathbb{C}$	NP-hard
Symmetric Singular Value over $\mathbb{R}$	NP-hard
Approximating Singular Vector over $\mathbb{R}, \mathbb{C}$	NP-hard
Spectral Norm over $\mathbb{R}$	NP-hard
Symmetric Spectral Norm over $\mathbb{R}$	NP-hard
Approximating Spectral Norm over $\mathbb{R}$	NP-hard
Nonnegative Definiteness	NP-hard
Best Rank-1 Approximation	NP-hard
Best Symmetric Rank-1 Approximation	NP-hard
Rank over $\mathbb{R}$ or $\mathbb{C}$	NP-hard
Enumerating Eigenvectors over $\mathbb{R}$	#P-hard
Combinatorial Hyperdeterminant	NP-, #P-, VNP-hard
Geometric Hyperdeterminant	Conjecturally NP-hard

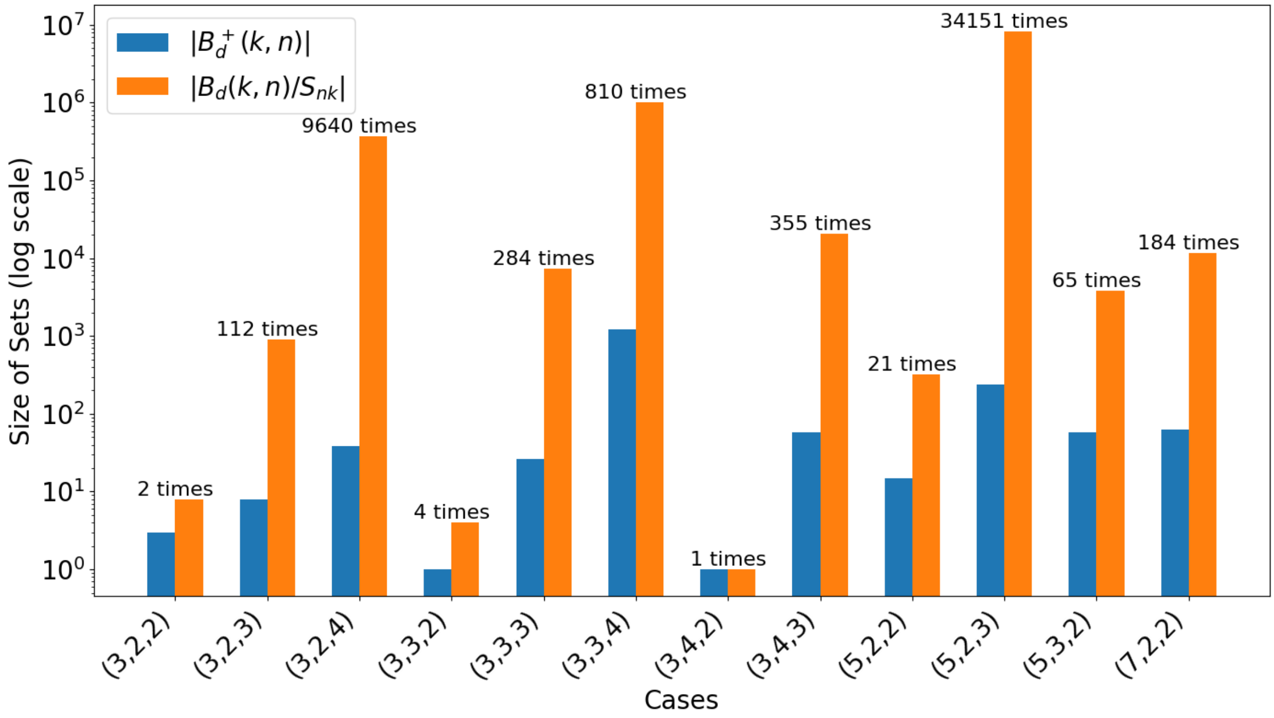

Figure 3.1 – Comparison of index spaces: lattice tables vs tables via symmetries. The x-axis represent the cases of (d, n, k) and the labels at the top of the bars represent the ratio $B_d^{lex}(k, n)/B_d^+(k, n)$.

Since there is no hope for improvement in (1), we tackle the computational challenge by (2). In particular, the size of the search reduced by considering spaces B^+ and A^+ instead of B^{lex} and A^{lex} spaces. Figure 3.1 and Figure 3.2 show that this offers an exponential improvement of a naive algorithm.

In this chapter this problem is addressed and the algorithm for description of invariant polynomials over tensor space $(\mathbb{C}^n)^{\otimes d}$ of minimal possible degree is constructed, thus providing a tool to distinguish orbits in most reasonable time. As it was shown in Theorem 2.3.2, the degree lower bound is $n\lceil n^{1/(d-1)}\rceil$, which is sharp.

3.1 Problem formulation

The algorithm described is based on the results of the previous chapter and is reported in the following siries of papers [71–73]. In particular, it requires the following propositions.

Proposition 3.1.1 (c.f. Corollary 2.3.16). *Let d be odd. If $T \in A(\lambda)$ has two equal columns then $\Delta_T = 0$.*

For $\boldsymbol{\lambda} = (n \times k)^{\times d}$ let $A_d(n, k) := A(\boldsymbol{\lambda})$, $B_d(k, n) := B(\boldsymbol{\lambda}')$, and analogically $A_d^+(n, k) := A^+(\boldsymbol{\lambda})$, $B_d^+(k, n) := B^+(\boldsymbol{\lambda}')$ and $A_d^{lex}(n, k) := A_d^{lex}(n, k), B_d^{lex}(k, n) := B_d(k, n)/S_{nk}$. Also, the weaker version of earlier statement is used.

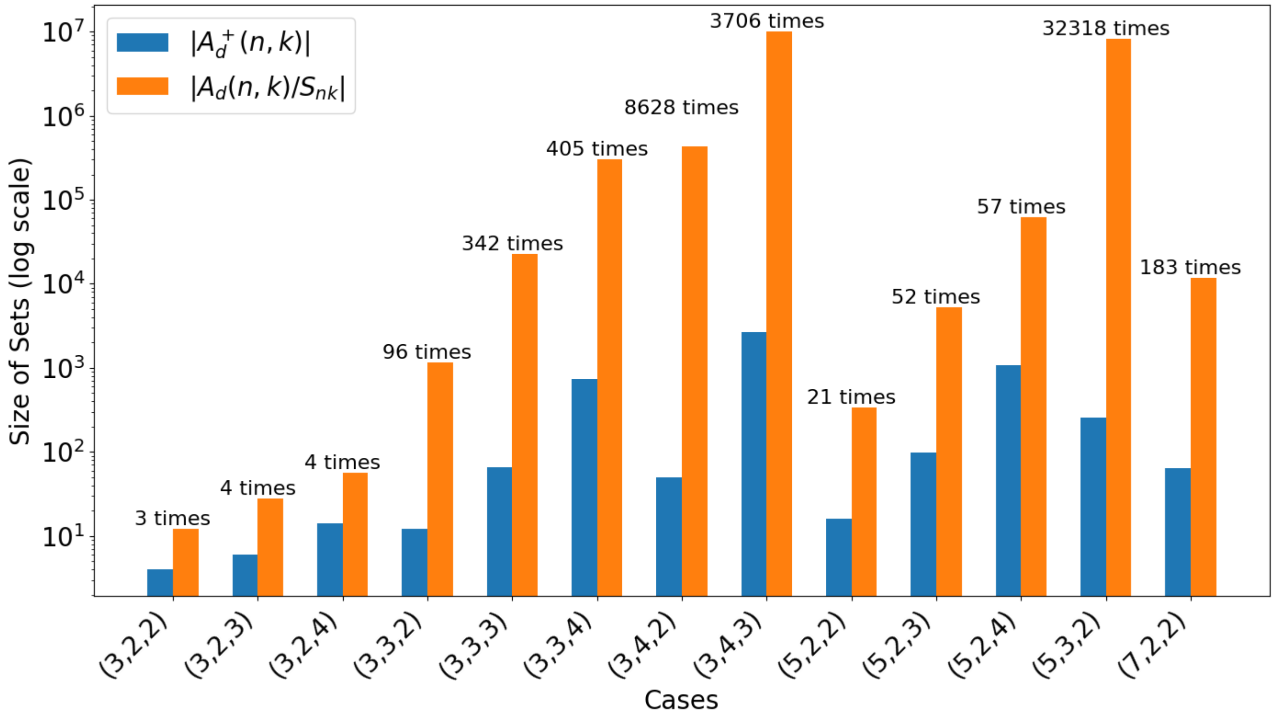

Figure 3.2 – Comparison of monomial spaces: lattice tables vs tables via symmetries. The x-axis represent the cases of (d, n, k) and the labels at the top of the bars represent the ratio $A_d^{lex}(k,n)/A_d^+(k,n)$

Proposition 3.1.2 (c.f. Proposition 2.3.5). *The polynomials* $\{\Delta_T\}$ *indexed by* $d \times m$ *tables* $T \in B_d^+(k,n)$ *span the space* $\mathbb{C}[(\mathbb{C}^n)^{\otimes d}]_{nk}^{\mathrm{SL}(n)^d}$ *of degree-nk invariant polynomials.*

Latter proposition implies, that it is enough to consider tables $T \in B_d^+(k,n)$ for suitable $k = \lceil n^{1/(d-1)} \rceil$ or $k = \lceil n^{1/(d-1)} \rceil + 1$. Recall the formula for Δ_T:

$$\begin{aligned}\Delta_T(X) &= \sum_{\sigma:[nk]\to[n]^d} \mathrm{sgn}_T(\sigma) \prod_{i=1}^{nk} X_{\sigma(i)} \\ &= \sum_{S\in A_d(n,k)} \mathrm{sgn}_T(S) \prod_{i=1}^{nk} X_{S(i)} \\ &= \sum_{S\in A_d^{lex}(n,k)} a(T,S) \prod_{i=1}^{nk} X_{S(i)}\end{aligned}$$

where $A_d^{lex}(n,k) = A_d^{lex}(n,k)$ is the set of representatives of orbits under the action of column permutation group S_{nk} and $a(T,S) \in \mathbb{Z}$ are coefficient at monomial x_S of Δ_T which can be described as follows:

$$a(T,S) := \frac{1}{\mathrm{stab}(S)} \sum_{\pi\in S_{nk}} \mathrm{sgn}_T(\pi S). \tag{3.1}$$

The effective algorithm is provided (more effective than naive algorithm) that computes these coefficients in the further sections.

3.2 Algorithm A': lattice words

A *lattice word* is string composed by positive integers, such that on every prefix contains at least as many integers i as integers $i+1$. A lattice word w is said to be of weight $\lambda \vdash m$ if the number of occurrences of i is λ_i. In this section the algorithm for generating a lattice words of specific length is provided.

a) Description of algorithm

Algorithm A'. Use simple backtracking algorithm that works as follows:

1 initially, let *word* be empty string, *cnt* length m array of 0s, λ be the vector corresponding to partition. Put letters one by one from left to right.

2 Assume at some point the *word* has length n and array $cnt[i]$ equals to the number of occurrences of i in *word*;

3 if the length of a word *word* is equal to m, then add *word* to the *result* (list of words);

4 otherwise iterate over $i = 1, \ldots, \ell(\lambda)$ and try to put the letter i at current position; this is possible if the number of is already put is less that the total number ($cnt[i] < \lambda[i]$) and the number of occurrences of $i-1$ is strictly greater ($cnt[i-1] > cnt[i]$);

5 if conditions for i met, put i at the end of *word*, increase $cnt[i]$ by 1 and recursively go to step 2;

6 after recursion is over, remove the last letter and decrease the counter $cnt[i]$ by 1 back.

The implementation of the algorithm in Python is provided in Listing 2. Here is a pseudocode of this algorithm.

Algorithm A' Lattice Words Generation

```
procedure GenerateLatticeWords(word, cnt, λ)
    if len(word) == ∑λ then
        result.append(word)
        return
    end if
    for i = 1 to len(λ) do
        if cnt[i] < λ[i] and (i == 1 or cnt[i − 1] > cnt[i]) then
            word.append(i)
            cnt[i] += 1
            GenerateLatticeWords(word, cnt, λ)
            cnt[i] −= 1
            word.pop()
        end if
    end for
end procedure
```

b) Analysis

Now estimate the time complexity of an algorithm.

Direct Enumeration: The algorithm's primary task is to enumerate every possible lattice word that can be formed while adhering to the restrictions imposed by λ and the lattice word condition. Each call to the recursive function corresponds to an extension of a word that is still potentially valid, and each valid extension at the maximum depth corresponds to one complete lattice word.

Complete Exploration: The algorithm explores every potential path in the search space that could yield a valid lattice word, backtracking when a particular path can no longer yield a valid word. Therefore, the time spent by the algorithm is proportional to the number of complete paths (or lattice words) it successfully constructs.

Efficiency and Pruning: The efficiency of the algorithm is not strictly tied to the operations at each step (which are generally $O(1)$ for adding to the word and checking conditions) but rather to how effectively it can prune paths that cannot result in valid lattice words. Each recursive call effectively generates a new "state" or partial word, and the depth of recursion (and thus the complexity) is directly linked to the number of these successful states.

Thus, **the time complexity of the algorithm is** $O(|A^+(\lambda)|)$, i.e. big-O of the number of lattice words of weight λ.

This complexity measure is particularly useful for understanding the practical performance of the algorithm because it directly relates to the output size. In cases where the constraints (i.e., the lattice condition and the frequency requirements from λ) severely restrict the number of valid words, the algorithm will inherently perform faster due to less branching in the recursive search tree. Conversely, when there are fewer constraints (e.g., when the parts of λ are relatively balanced or small), the number of valid lattice words — and hence the complexity — increases.

3.3 Algorithm A: lattice tables

The implementation for generating lattice tables ($A_d^+(n,k)$ or $B_d^+(k,n)$) is provided in the Listing 3 that is called **Algorithm A**, which is generalization of Algorithm A'. It is described below.

Algorithm A Lattice Tables Generation

```
procedure GenMagicCubesRec(n, k, id, cur, cnt, all, no_rep)
    if len(cur) == n × k then
        all.append(cur)
        return
    end if
    if id == len(CUBE) then
        return
    end if
    skip ← False
    for i ← 0 to d − 1 do
        if cnt[i][CUBE[id][i]] == n then
            skip ← True
            break
        end if
    end for
    for i ← 0 to d − 1 do
        xi ← CUBE[id][i]
        if xi > 0 and cnt[i][xi − 1] == cnt[i][xi] then
            skip ← True
            break
        end if
    end for
    if not skip then
        cur.append(CUBE[id])
        for i ← 0 to d − 1 do
            cnt[i][CUBE[id][i]] += 1
        end for
        GenMagicCubesRec(n, k, id + no_rep, cur, cnt, all, no_rep)
        cur.pop_back()
        for i ← 0 to d − 1 do
            cnt[i][CUBE[id][i]] −= 1
        end for
    end if
    GenMagicCubesRec(n, k, id + 1, cur, cnt, all, no_rep)
end procedure
```

The `genMagicCubesRec` function is a recursive algorithm designed to generate $d \times m$ tables, where each row is a lattice word, and the occurrence of each number from 1 to k in a row is exactly n. The tables are represented as vectors of points (d-tuples), with each point corresponding to a column in the table. Here's a breakdown of the algorithm:

a) Parameters

- n: The number of occurrences of each number (1 through k) in a row.

- k: Specifies the range of numbers each element in a point can take, from 1 to k.

- id: The current index in the CUBE vector that is being considered.

- cur: Current vector of points that are already included in the table being constructed.

- cnt: A matrix tracking the number of occurrences of each number (1 through k) in each dimension of the current table.

- all: A vector of all successfully generated tables.

- no_rep: A boolean flag that, when true, disallows repetition of columns (points) in the table.

b) Process

1 *Base Case:* If the size of the current table (cur) equals $n \times k$, a complete table has been constructed. This table is then added to the all vector, and the function returns to explore other possibilities.

2 *Termination Point:* If id equals the size of the CUBE (i.e., all points have been considered without success), the function returns without making further recursive calls.

3 *Skipping Points:*

- The algorithm first checks if adding the current point (CUBE[id]) would exceed the allowable count (n) of any number in any dimension. If so, this point is skipped.

- It also ensures that each row corresponds to a lattice word. A lattice word requires that no number x_i in dimension i appears unless all numbers less than x_i have appeared n times in that dimension. If this condition is violated, the point is skipped.

4 *Recursive Inclusion:*

- If the point isn't skipped, it is tentatively added to the cur vector.

- The counts (cnt) are updated to reflect this addition.

- A recursive call is made to attempt adding the next point, with id incremented by 1 if repetitions (no_rep) are not allowed, ensuring each column is unique.

- After the recursive call, the point is removed from cur, and the counts are decremented, effectively backtracking.

5 *Continue Exploration:*

- Regardless of whether the current point was added or skipped, a recursive call is made to consider the next point in the CUBE vector ($\mathrm{id} + 1$), allowing the algorithm to explore all possible configurations.

c) Outcome

The function effectively generates all possible $d \times m$ tables that meet the given criteria, storing each valid table in the all vector. The use of recursion and backtracking ensures that all configurations are explored while maintaining the constraints imposed by lattice words and the occurrence counts of each number.

Results of an Algorithm A are displayed in Table 3.4. The comparative analysis can be found in Table 3.2 and Table 3.3. Also, efficiency of consideration of lattice tables over lex-ordered tables were displayed in Figure 3.2 and Figure 3.1.

Table 3.2 – Reduction Efficiency Table for Sets A and A^+ for monomial space.

Cases (d, n, k)	**Size of** A^{lex}	**Size of** A^+	**Ratio** (A^{lex}/A^+)	**Reduction (%)**
(3,2,2)	12	4	3.000	66.667
(3,2,3)	28	6	4.667	78.571
(3,2,4)	57	14	4.071	75.439
(3,3,2)	1152	12	96.000	98.958
(3,3,3)	22620	66	342.727	99.708
(3,3,4)	302274	746	405.190	99.753
(3,4,2)	431424	50	8628.480	99.988
(3,4,3)	10000000	2698	3707.263	99.973
(5,2,2)	336	16	21.000	95.238
(5,2,3)	5200	99	52.525	98.096
(5,3,2)	8241264	255	32313.388	99.990
(7,2,2)	11712	64	183.000	99.455

Table 3.3 – Reduction Efficiency Table for Sets B and B^+ for index space.

Cases (d, n, k)	**Size of** B^{lex}	**Size of** B^+	**Ratio** B^{lex}/B^+	**Reduction (%)**
(3,2,2)	8	3	2.667	62.500
(3,2,3)	900	8	112.500	99.111
(3,2,4)	366336	38	9640.421	99.990
(3,3,2)	4	1	4.000	75.000
(3,3,3)	7392	26	284.308	99.648
(3,3,4)	1000000	1234	810.211	99.877
(3,4,2)	1	1	1.000	0.000
(3,4,3)	20619	58	355.500	99.718
(5,2,2)	320	15	21.333	95.313
(5,2,3)	8162208	239	34150.239	99.997
(5,3,2)	3824	58	65.931	98.439
(7,2,2)	11648	63	184.095	99.460

3.4 Algorithm B: calculating polynomial coefficients

The Algorithm described in this subsection is called **Algorithm B**.

The code, provided in Listing 4 includes two primary functions intended to compute the coefficient $a(T, S)$ as defined in the mathematical formula (3.1). These functions are implemented to utilize mathematical properties of permutations and group theory to optimize the computation. Below is a detailed explanation of each part of the code:

Algorithm B Calculating Polynomial Coefficients

```
function GET_GST(s, t)
    Ys ← get_young_subgroup(s)                    ▷ Stabilizer of word s
    Yt ← get_young_subgroup(t)                    ▷ Stabilizer of word t
    h ← get_nonzero_shift(s, t)                   ▷ Ensure that sgn_s(π · t) ≠ 0
    res ← Ys × {h} × Yt                           ▷ Set product
    return res
end function
function GET_ATS(T, S)
    stab ← get_stab(S)                            ▷ Stabilizer of table S
    doubleCoset ← GetGST(T[0], S[0])              ▷ Double coset of first rows of tables
    ats ← 0
    for each p in doubleCoset do
        x ← sgn(T, p · S)
        ats += x
    end for
    ats /= stab
    return ats
end function
```

Function: `get_GST(word s, word t)`

Purpose: This function generates a set of permutations that significantly reduces the complexity of the problem by focusing only on permutations that contribute non-zero terms to the sum in the calculation of $a(T, S)$.

Steps:

1 **Young Subgroups**: Calculates the Young subgroups `Ys` and `Yt` for words `s` and `t`, respectively. Young subgroups are stabilizers in the symmetric group, which partition the group into classes simplifying the computation of permutation properties.

2 **Non-zero Shift**: Determines a permutation `h` via `get_nonzero_shift(s, t)` such that the sign condition $\mathrm{sgn}_s(\pi \cdot t) \neq 0$ is satisfied. This permutation ensures that subsequent computations consider only effective permutations.

3 **Set Product**: Computes the set product of `Ys`, {`h`}, and `Yt`, representing the required permutations for computing the sign as per the formula, optimizing the overall process.

Function: `get_ats(table T, table S)`

Purpose: Computes the coefficient $a(T, S)$ using the permutations generated by `get_GST`, designed to operate efficiently by reducing the computation space.

Steps:

1 **Stabilizer**: Computes the stabilizer `stab` of table `S`, a group that keeps `S` invariant, used as a divisor in the final computation for normalization.

2 **Double Coset**: Retrieves the double coset using `get_GST(T[0], S[0])`, focusing on permutations involving the first rows `T[0]` and `S[0]` of tables `T` and `S`. This reduces the problem size significantly, offering computational savings.

3 **Summation and Sign Calculation**: Iterates over permutations in the double coset, calculating the sign of `T` when permutation `p` is applied to `S` and summing the results.

4 **Normalization**: The sum `ats` is then divided by the stabilizer `stab` to obtain the final coefficient.

The approach used in the code significantly reduces computational complexity by leveraging properties of permutations and group theory. The main computational savings come from reducing the full factorial search space of $(nk)!$ to a much smaller space defined by specific group properties. This is achieved as follows:

- **Double Coset**: The calculation reduces the need to consider all permutations of S_{nk}, which has a size of $(nk)!$. Instead, by focusing on the double coset of the form $Y_{T_i}\omega Y_{S_i}$, the space is narrowed to permutations where $\pi \in Y_{T_i}\omega Y_{S_i}$ for some $\omega \in S_{nk}$. This subset is much smaller because it is restricted to permutations that maintain the structure defined by Young subgroups.

- **Size of the Reduced Space**: The size of this reduced space can be quantified as $O((n!)^{2k})$ instead of naive $O((nk)!)$. This represents a substantial reduction, especially for large n and k, making the computation feasible even for larger tables and higher dimensions where factorial growth would otherwise make the computation intractable.

This optimized approach allows the computation of the coefficient $a(T,S)$ to be practical by focusing only on permutations that contribute non-zero values to the sum, thus eliminating a vast number of unnecessary operations and checks. The use of group theory not only simplifies the computational workload but also ensures that the calculations are mathematically robust, leveraging the symmetries and properties inherent in the structure of permutation groups.

3.5 Algorithm for invariant polynomials

In this section the main algorithm to construct invariant polynomials of minimal degree (or of any degree) is provided, utilizing algorithms A and B.

3.5.1 Algorithm C: invariant polynomials

Algorithm C is designed to compute invariant polynomials of minimal degree. This algorithm aims to produce a basis of the vector space $\mathbb{C}[(\mathbb{C}^n)^{\otimes d}]_{nk}$, where $k = \delta_d(n)$. In the best case scenario, the algorithm outputs a complete basis for the space, while in the least favorable case, it provides a set of linearly independent polynomials. Below one can find pseudocode for the algorithm C.

Algorithm C Computation of basis of invariant polynomials of given degree

```
function ComputeBasis(d, n)
    k ← δ_d(n)
    A ← GenerateSetA(d, n, k)                    ▷ Using Algorithm A
    B ← GenerateSetB(d, k, n)                    ▷ Using Algorithm A
    M ← InitializeEmptyMatrix()                  ▷ Matrix to store coefficients a(T, S)
    for each T in B do
        for each S in A do
            a_{T,S} ← ComputeCoefficient(T, S)   ▷ Using Algorithm B
            AddCoefficientToMatrix(M, a_{T,S})
        end for
    end for
    𝒯 ← GaussJordanElimination(M)                ▷ Compute basis of row space
    return 𝒯
end function
```

a) Description

Input: An odd integer d and a dimension n.

Output: Set of tables from $B_d^+(k,n)$ that form the basis of the vector space $\mathbb{C}[(\mathbb{C}^n)^{\otimes d}]_{nk}$, where $k = \delta_d(n)$.

Steps:

1 Begin by generating the sets $A_d^+(n,k)$ and $B_d^+(k,n)$ using Algorithm A. These sets are crucial as they define the landscape over which our algorithm will operate, containing elements that potentially contribute to the basis of our target vector space.

2 Then iterate over $d \times m$ tables T contained within the set $B_d^+(k,n)$. Each table T represents a candidate tensor product structure that could be part of the basis.

3 For each table T, iterate over $d \times m$ tables S from the set $A_d^+(n,k)$. This step is designed to explore potential interactions between elements of $A_d^+(n,k)$ and the current table T under consideration.

4 Compute the coefficient $a(T, S)$ for each pair of tables (T, S). This coefficient is a crucial metric that quantifies the degree of symmetry or interaction between the tensors represented by T and S.
5 Employ Gauss-Jordan elimination to compute the basis of the row space of the matrix formed by coefficients $a(T, S)$. Let $\mathcal{T}$ be the set of rows that constitute a basis. This step is pivotal as it determines the linear independence and spans of potential basis vectors.
6 The tables corresponding to $\mathcal{T}$ then represent the basis of the space, thus concluding the algorithm.

b) Computational Complexity

The computational complexity of this algorithm is primarily determined at the matrix operation stages (Steps 5 and 6). Ordinary Gauss-Jordan elimination on an $n \times m$ matrix A operates in $O(nm \cdot \mathrm{rank}(A))$ time. Therefore, the overall complexity of the algorithm is $O(|B_d^+(k, n)| \cdot |A_d^+(n, k)| \cdot g_d(n, k))$. Since $g_d(n, k)$ is relatively smaller compared to the sizes of $|B_d^+(k, n)|$ and $|A_d^+(n, k)|$, the computation remains feasible for several small instances.

c) Discussion

Iterating over the reduced space $A_d^+(n, k)$ instead of the full space $A_d^{lex}(n, k)$ represents a significant optimization. By focusing on $A_d^+(n, k)$, the size of the monomial space is reduced dramatically, thereby accelerating the search process at the potential cost of losing some information about the independence of the polynomials. However, this trade-off is often beneficial as it speeds up the process dramatically without substantially compromising the quality of the results. This optimization is supported by the following lemma from linear algebra, which confirms that if a subset of rows spans the row space of a submatrix, these rows also span the row space of the entire matrix.

Lemma 3.5.1. *Let A be an $n \times m$ matrix, and let J be a subset of columns of A forming a submatrix A_J. Suppose that the rank of A_J is equal to the rank of A, denoted as r. If I is a subset of rows of size r that span the row space of A_J, then the rows in I also span the row space of A.*

In practice, this lemma allows us to replace the full monomial space $A_d^{lex}(n, k)$ with $A_d^+(n, k)$ during the algorithm execution, ensuring efficiency while still producing a robust set of linearly independent, and potentially basis-forming polynomials. There are representation theoretic reasons to consider only the set $A_d^+(n, k)$ that is going to be address elsewhere. In all examples below the monomial space $A_d^+(n, k)$ instead of $|A_d^{lex}(n, k)$ is considered, motivated by the Table 3.4.

Table 3.4 – Comparison of the size of the sets $A_d^{lex}(n,k)$ and $A_d^+(n,k)$, $B_d^{lex}(k,n)$ and $B_d^+(k,n)$.

(d,n,k)	$\lvert A_d^+(n,k)\rvert$	$\lvert A_d^{lex}(n,k)\rvert$	$\lvert B_d^+(k,n)\rvert$	$\lvert B_d^{lex}(k,n)\rvert$
$(3,2,2)$	4	12	3	8
$(3,2,3)$	6	28	8	900
$(3,2,4)$	14	57	38	366336
$(3,3,2)$	12	1152	1	4
$(3,3,3)$	66	22620	26	7392
$(3,3,4)$	746	302274	1234	$> 10^6$
$(3,4,2)$	50	431424	1	1
$(3,4,3)$	2698	$> 10^7$	58	20619
$(5,2,2)$	16	336	15	320
$(5,2,3)$	99	5200	239	8162208
$(5,2,4)$	1086	61992	?	?
$(5,3,2)$	255	8241264	58	3824
$(7,2,2)$	64	11712	63	11648

3.6 Computation of special cases

In this section computational results of Algorithm C are presented. The basis of invariant polynomials of small degrees in each of the listed cases is found.

3.6.1 3 qubits

Let $d = 3, n = 2, k = 2$. According Table A.5 in Appendix B the dimension of the space of degree-4 and degree-6 invariant polynomials in $\mathbb{C}[(\mathbb{C}^2)^{\otimes 3}]^{\mathrm{SL}(2)^3}$ is:

$$\dim \mathbb{C}[(\mathbb{C}^2)^{\otimes 3}]_4^{\mathrm{SL}(2)^3} = g_3(2,2) = 1$$

There are 4 tables in $A_d^+(n,k)$ and 3 tables in $B_d^+(k,n)$. Among the latter, the following unique table is found corresponding to unique invariant polynomial of degree 4:

$$T = \begin{pmatrix} 0011 \\ 0011 \\ 0101 \end{pmatrix} \tag{3.2}$$

which corresponds to fundamental invariant $\det_{2\times 2\times 2}$, which is unique generator of corresponding ring.

3.6.2 3 qutrits

Let $d = 3, n = 3, k = 2$. According Table A.5 in Appendix B the dimension of the space of degree-6 and degree-9 invariant polynomials in $\mathbb{C}[(\mathbb{C}^2)^{\otimes 3}]^{\mathrm{SL}(2)^3}$ is:

$$\dim \mathbb{C}[(\mathbb{C}^3)^{\otimes 3}]_6^{\mathrm{SL}(2)^3} = g_3(3,2) = 1$$
$$\dim \mathbb{C}[(\mathbb{C}^3)^{\otimes 3}]_9^{\mathrm{SL}(2)^3} = g_3(3,3) = 1$$

There are 66 tables in $A_d^+(n,k)$ and 26 tables in $B_d^+(k,n)$. Among the latter, the following unique tables are found which correspond to unique invariant polynomials of degree 6 and 9:

$$T_1 = \begin{pmatrix} 000111 \\ 001011 \\ 010101 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 111222333 \\ 112223331 \\ 122233311 \end{pmatrix} \tag{3.3}$$

3.6.3 5 qubits

Let $d = 5, n = 2, k = 2$. According Table A.6 in Appendix B the dimension of the space of degree-4 and degree-6 invariant polynomials in $\mathbb{C}[(\mathbb{C}^2)^{\otimes 5}]^{\mathrm{SL}(2)^5}$ is:

$$\dim \mathbb{C}[(\mathbb{C}^2)^{\otimes 5}]_4^{\mathrm{SL}(2)^5} = g_5(2,2) = 5$$
$$\dim \mathbb{C}[(\mathbb{C}^2)^{\otimes 5}]_6^{\mathrm{SL}(2)^5} = g_5(2,3) = 1.$$

There are 16 tables in $A_d^+(n,k)$ and 15 tables in $B_d^+(k,n)$. Among the latter, the following 5 tables are found, corresponding to basis of invariant polynomials of degree 4:

$$T_1 = \begin{pmatrix} 0011 \\ 0011 \\ 0011 \\ 0011 \\ 0101 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0011 \\ 0011 \\ 0011 \\ 0101 \\ 0101 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0011 \\ 0011 \\ 0011 \\ 0101 \\ 0011 \end{pmatrix},$$

$$T_4 = \begin{pmatrix} 0011 \\ 0011 \\ 0101 \\ 0011 \\ 0101 \end{pmatrix}, \quad T_5 = \begin{pmatrix} 0011 \\ 0011 \\ 0101 \\ 0101 \\ 0011 \end{pmatrix}$$

that have 192 terms in total and 32 terms corresponding to $A_5^+(2,2)$. For instance,

$$\begin{aligned}\Delta_{T_1}(X) &= X_{11111}^2 X_{22222}^2 \\ &- 2(X_{11111}X_{11112}X_{22221}X_{22222} + X_{11111}X_{11121}X_{22212}X_{22222} + X_{11111}X_{11122}X_{22211}X_{22222} \\ &+ X_{11111}X_{11211}X_{22122}X_{22222} + X_{11111}X_{11212}X_{22121}X_{22222} + X_{11111}X_{12222}X_{21111}X_{22222}) \\ &+ 2(X_{11111}X_{11221}X_{22112}X_{22222} + X_{11111}X_{11222}X_{22111}X_{22222} + X_{11111}X_{12111}X_{21222}X_{22222} \\ &+ X_{11111}X_{12112}X_{21221}X_{22222} + X_{11111}X_{12121}X_{21212}X_{22222} + X_{11111}X_{12221}X_{21112}X_{22222} \\ &+ X_{11111}X_{12122}X_{21211}X_{22222} + X_{11111}X_{12211}X_{21122}X_{22222} + X_{11111}X_{12212}X_{21121}X_{22222}) + \ldots\end{aligned}$$

here 32 terms from $A_5^+(2,2)$ are displayed. Each of these polynomials does not null at unit tensor.

Also, for $(d,n,k) = (5,2,3)$ invariant polynomial of degree 6 is obtained:

$$T_1 = \begin{pmatrix} 001122 \\ 001212 \\ 010122 \\ 010212 \\ 012012 \end{pmatrix}$$

which is unique up to a scale in the space $\mathbb{C}[(\mathbb{C}^2)^{\otimes 5}]_6^{\mathrm{SL}(2)^5}$. There are 99 tables in $|A_d^+(n,k)|$ and 239 in $|B_d^+(n,k)|$.

3.6.4 5 qutrits

Let $d = 5, n = 3, k = 2$. According Table A.6 in Appendix B the dimension of the space of degree-6 invariant polynomials $\mathbb{C}[(\mathbb{C}^3)^{\otimes 5}]_6^{\mathrm{SL}(3)^5}$ is:

$$\dim \mathbb{C}[(\mathbb{C}^3)^{\otimes 5}]_6^{\mathrm{SL}(3)^5} = g_5(3,2) = 11.$$

There are 255 tables in $A_d^+(n,k)$ and 58 tables in $B_d^+(k,n)$. Among the latter, by Algorithm C the following 11 tables corresponding to basis of invariant polynomials of degree 6 is found:

$$T_1 = \begin{pmatrix} 000111 \\ 001011 \\ 001101 \\ 001101 \\ 010011 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 000111 \\ 001011 \\ 001101 \\ 010011 \\ 001101 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 000111 \\ 001011 \\ 001101 \\ 010011 \\ 010011 \end{pmatrix}, \quad T_4 = \begin{pmatrix} 000111 \\ 001011 \\ 001101 \\ 010011 \\ 010101 \end{pmatrix},$$

$$T_5 = \begin{pmatrix} 000111 \\ 001011 \\ 001101 \\ 010011 \\ 001011 \end{pmatrix}, \quad T_6 = \begin{pmatrix} 000111 \\ 001011 \\ 001101 \\ 001101 \\ 010101 \end{pmatrix}, \quad T_7 = \begin{pmatrix} 000111 \\ 001011 \\ 001101 \\ 010011 \\ 000111 \end{pmatrix}, \quad T_8 = \begin{pmatrix} 000111 \\ 001011 \\ 001101 \\ 010101 \\ 000111 \end{pmatrix},$$

$$T_9 = \begin{pmatrix} 000111 \\ 001011 \\ 001101 \\ 000111 \\ 010101 \end{pmatrix}, \quad T_{10} = \begin{pmatrix} 000111 \\ 001011 \\ 001101 \\ 001011 \\ 010101 \end{pmatrix}, \quad T_{11} = \begin{pmatrix} 000111 \\ 001011 \\ 001101 \\ 010101 \\ 001011 \end{pmatrix}$$

3.6.5 7 qubits

Let $d = 7, n = 2, k = 2$. According Table A.7 in Appendix B the dimension of the space of degree-4 invariant polynomials $\mathbb{C}[(\mathbb{C}^2)^{\otimes 7}]_4^{\mathrm{SL}(2)^7}$ is:

$$\dim \mathbb{C}[(\mathbb{C}^2)^{\otimes 7}]_4^{\mathrm{SL}(2)^7} = g_7(2,2) = 21$$

There are 64 tables in $A_d^+(n,k)$ and 63 tables in $B_d^+(k,n)$. Among the latter, the following 21 tables corresponding to basis of invariant polynomials of degree 4 is found:

$$T_1 = \begin{pmatrix} 0011 \\ 0011 \\ 0011 \\ 0011 \\ 0011 \\ 0011 \\ 0101 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0011 \\ 0011 \\ 0011 \\ 0011 \\ 0011 \\ 0101 \\ 0101 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0011 \\ 0011 \\ 0011 \\ 0011 \\ 0011 \\ 0101 \\ 0011 \end{pmatrix}, \quad T_4 = \begin{pmatrix} 0011 \\ 0011 \\ 0011 \\ 0011 \\ 0101 \\ 0011 \\ 0101 \end{pmatrix},$$

$$T_5 = \begin{pmatrix} 0011 \\ 0011 \\ 0011 \\ 0011 \\ 0101 \\ 0101 \\ 0011 \end{pmatrix}, \quad T_6 = \begin{pmatrix} 0011 \\ 0011 \\ 0011 \\ 0101 \\ 0011 \\ 0011 \\ 0101 \end{pmatrix}, \quad T_7 = \begin{pmatrix} 0011 \\ 0011 \\ 0011 \\ 0101 \\ 0101 \\ 0101 \\ 0011 \end{pmatrix}, \quad T_8 = \begin{pmatrix} 0011 \\ 0011 \\ 0101 \\ 0011 \\ 0011 \\ 0011 \\ 0101 \end{pmatrix},$$

$$T_9 = \begin{pmatrix} 0011 \\ 0011 \\ 0101 \\ 0101 \\ 0101 \\ 0101 \\ 0011 \end{pmatrix}, \quad T_{10} = \begin{pmatrix} 0011 \\ 0101 \\ 0011 \\ 0011 \\ 0011 \\ 0011 \\ 0101 \end{pmatrix}, \quad T_{11} = \begin{pmatrix} 0011 \\ 0101 \\ 0101 \\ 0101 \\ 0101 \\ 0101 \\ 0011 \end{pmatrix}, \quad T_{12} = \begin{pmatrix} 0011 \\ 0101 \\ 0011 \\ 0101 \\ 0101 \\ 0101 \\ 0011 \end{pmatrix},$$

$$T_{13} = \begin{pmatrix} 0011 \\ 0101 \\ 0101 \\ 0011 \\ 0011 \\ 0011 \\ 0101 \end{pmatrix}, \quad T_{14} = \begin{pmatrix} 0011 \\ 0011 \\ 0101 \\ 0011 \\ 0101 \\ 0101 \\ 0011 \end{pmatrix}, \quad T_{15} = \begin{pmatrix} 0011 \\ 0101 \\ 0101 \\ 0011 \\ 0101 \\ 0101 \\ 0011 \end{pmatrix}, \quad T_{16} = \begin{pmatrix} 0011 \\ 0101 \\ 0011 \\ 0101 \\ 0011 \\ 0011 \\ 0101 \end{pmatrix},$$

$$T_{17} = \begin{pmatrix} 0011 \\ 0011 \\ 0101 \\ 0101 \\ 0011 \\ 0011 \\ 0101 \end{pmatrix}, \quad T_{18} = \begin{pmatrix} 0011 \\ 0101 \\ 0011 \\ 0011 \\ 0101 \\ 0101 \\ 0011 \end{pmatrix}, \quad T_{19} = \begin{pmatrix} 0011 \\ 0101 \\ 0101 \\ 0101 \\ 0011 \\ 0011 \\ 0101 \end{pmatrix}, \quad T_{20} = \begin{pmatrix} 0011 \\ 0011 \\ 0011 \\ 0101 \\ 0011 \\ 0101 \\ 0011 \end{pmatrix},$$

$$T_{21} = \begin{pmatrix} 0011 \\ 0011 \\ 0101 \\ 0101 \\ 0011 \\ 0101 \\ 0011 \end{pmatrix}.$$

CONCLUSION

The results obtained are important fundamental results in such fields as representation theory, tensor theory and quantum information theory.

Summary of Dissertation Research Results. This dissertation presents the development of theoretical frameworks and algorithms for understanding and applying tensor invariants to quantum information theory. The primary focus is on establishing a systematic approach to generating and utilizing tensor invariants, particularly in the context of quantum entanglement and geometric complexity theory. The research outcomes include:

1 Explored the fundamental properties of tensor invariants within the framework of Schur-Weyl duality.Extended the analysis to higher-dimensional tensor spaces, deriving a comprehensive understanding of their mathematical structures and symmetries.

2 Developed mathematical models for invariant polynomials in tensor spaces, identifying the smallest degree invariants and their generating sets. Introduced algorithms for generating invariant polynomials, providing concrete examples and properties.

3 Proposed a hierarchical structure for studying tensor invariants using partitions, highest weight vectors, and representation theory. Analyzed the role of Kronecker coefficients in understanding the complexity and invariance properties of tensors.

4 Developed and implemented algorithms for calculating polynomial coefficients and generating invariant polynomials. Conducted empirical experiments using C++ and Sage to validate the theoretical findings.

5 Applied the developed theoretical frameworks and algorithms to study quantum entanglement, demonstrating the practical significance of tensor invariants in this domain. Explored specific cases involving qubits and qutrits, providing insights into the entanglement properties of quantum states.

Assessment of Task Completeness. The research successfully addressed the following tasks:

1 Conducted a thorough analysis of the properties and applications of tensor invariants in quantum information theory.

2 Identified key mathematical structures and developed models for invariant polynomials.

3 Established algorithms for generating and applying tensor invariants.

4 Validated the theoretical findings through empirical testing and practical applications.

5 Explored the implications of the research in quantum information theory, particularly in the context of entanglement and geometric complexity.

Recommendations and Practical Applications. The practical significance of the research is multifaceted:

1 The developed algorithms and models can be integrated into quantum computing frameworks to enhance the understanding and measurement of quantum entanglement.
2 The research provides a robust mathematical foundation for studying tensor invariants, facilitating further advancements in both theoretical and applied quantum information theory.

Scientific Contribution and Comparison with Best Achievements. The following new results were achieved based on the research:

1 Developed a systematic approach to studying and generating tensor invariants, particularly for minimal degree cases.
2 Provided a comprehensive analysis of Kronecker coefficients and their role in tensor invariants.
3 Proposed algorithms that bridge the gap between theoretical mathematics and practical quantum computing applications.
4 Demonstrated the practical applications of tensor invariants in understanding and measuring quantum entanglement.

The research findings have been validated through publications and presentations in various international forums, ensuring their relevance and applicability in the field of quantum information theory. The results contribute significantly to the scientific understanding of tensor invariants and their practical applications, positioning the work at the forefront of research in this area.

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Appendix A TABLES OF DEGREE SEQUENCES

Following tables where calculated with Sage software [74]. The following code has being used to calculate the values of Kronecker coefficients. It uses theory of symmetric functions to calculate Kronecker coefficients, in particular internal product of Schur functions can be decomposed into the Schur basis [75] and the coefficients are the Kronecker coefficients:

$$s_\lambda * s_\mu = \sum_\nu g(\lambda, \mu, \nu) s_\nu.$$

This internal product is an expensive operation. To speed up calculation for rectangular Kronecker coefficients for $d > 3$ we use binary exponentiation: for $d = 5$ we compute $s_{n\times k}^{*2}$ and $s_{n\times k}^{*4}$, similarly for $d = 7$. This way, for $d = 5$ instead of 4 multiplications we use 3. See Listing 1.

Table A.1 – The table of $g_d(n, 2) = \dim \mathrm{Inv}_d(n)_{2n}$ for $0 \leqslant n \leqslant 16$, $d = 3, 5, 7$. Note that $g_d(n, 2) = g_d(2^{d-1} - n, 2)$ and $g_d(n, 2) = 0$ for $n > 2^{d-1}$.

$n\backslash d$	3	5	7
0	1	1	1
1	1	1	1
2	1	5	21
3	1	11	161
4	1	35	3341
5	0	52	64799
6	0	112	1407329
7	0	130	27536390
8	0	166	482181504
9	0	130	7403718609
10	0	112	99468725538
11	0	52	1168191022248
12	0	35	12009002387858
13	0	11	108266717444858
14	0	5	857991447205123
15	0	1	5991301282600760
16	0	1	36953889463653995

Table A.2 – The table of $\delta_3(n)/n$ for $1 \leqslant n \leqslant 16$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\delta_3(n)/n$	1	2	2	2	3	3	4	3	3	4	4	4	4	4	4	4

Table A.3 – The values of $\delta_d(n)/n$ for various d and n

$d \setminus n$	1	2	3	4	5	6	7	8	9	10
3	2	2	2	2	3	3	4	3	3	3
5	2	2	2	2	2	2	2	2	2	2
7	2	2	2	2	2	2	2	2	2	2

Table A.4 – The table of $g_d(n,3) = \dim \mathrm{Inv}_d(n)_{3n}$ for $0 \leqslant n \leqslant 9$, $d = 3,5,7$. Note that $g_d(n,3) = g_d(3^{d-1} - n, n)$ and $g_d(n,3) = 0$ for $n > 3^{d-1}$.

$n \backslash d$	3	5	7
0	1	1	1
1	1	1	1
2	0	1	70
3	1	385	636177
4	1	44430	9379255543
5	1	5942330	215546990657498
6	1	781763535	6136455833113627910
7	0	93642949102	191473697724924688999920
8	1	9856162505065	6100591257296003780834337810
9	1	894587378523908	190121112332748795911599731191284

Table A.5 – The table of $g_3(n,k) = \dim \mathrm{Inv}_3(n)_{kn} = g(n \times k, n \times k, n \times k)$ for $1 \leqslant k \leqslant 6$, $1 \leqslant n \leqslant 8$. Note that $g_3(n,k) = g_3(k^2 - n, k)$ and $g_3(n,k) = 0$ for $n > k^2$.

$k \backslash n$	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0
2	1	1	1	1	0	0	0	0
3	1	0	1	1	1	1	0	1
4	1	1	2	5	6	13	14	18
5	1	0	1	4	21	158	1456	9854
6	1	1	3	16	216	9309	438744	17957625

Table A.6 – The table of $g_5(n,k) = \dim \mathrm{Inv}_5(n)_{kn}$ for $1 \leqslant k \leqslant 5$, $1 \leqslant n \leqslant 6$.

$k \backslash n$	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	5	11	35	52	112
3	1	1	385	44430	5942330	781763535
4	1	36	44522	381857353	5219755745322	87252488565829772
5	1	15	6008140	5220537438711	10916817688177999825	36929519748583464067841925

Table A.7 – The table of $g_7(n,k) = \dim \mathrm{Inv}_7(n)_{kn}$ for $1 \leqslant k \leqslant 5$, $1 \leqslant n \leqslant 5$.

$k \backslash n$	1	2	3	4	5
1	1	0	0	0	0
2	1	21	161	3341	64799
3	1	70	636177	9379255543	215546990657498
4	1	3362	9379321798	220746106806871065	14446465578705208466014240
5	1	62204	215601786541974	14446471715159302533654142	5370640146091973101847897273759375505

Table A.8 – The table of $g_9(n,k) = \dim \mathrm{Inv}_9(n)_{kn}$ for $1 \leqslant k \leqslant 5$, $1 \leqslant n \leqslant 4$.

$k \backslash n$	1	2	3	4
1	1	0	0	0
2	1	85	3151	538525
3	1	2331	1120708625	2001892792552290
4	1	538882	2001892904676744	127405523125364290107394265
5	1	112063889	7776236632415272731072	39943390428604984740410365287203655408

Appendix B Listings

This chapter contains C++ implementations of algorithms introduced above.

```
def g(d, n, k):
  H = h([n*k])
  sh = [k]*n
  shur = s(sh)
  R = shur
  for i in range(0,d-1):
      R = shur.kronecker_product(R)
  res = H.scalar(R)
  print("d =", d, "n =", n, "k =", k, "dim =", res)
  return res

def g5(n, k):
  d = 5
  sh = [k]*n
  shur = s(sh)
  R = shur
  R2 = R.kronecker_product(R)
  R4 = R2.kronecker_product(R2)
  res = R4.scalar(R)
  print("d =", d, "n =", n, "k =", k, "dim =", res)
  return res

def g7(n, k):
  d = 7
  H = h([n*k])
  sh = [k]*n
  shur = s(sh)
  R = shur
  R2 = R.kronecker_product(R)
  R4 = R2.kronecker_product(R2)
  RR = R4.kronecker_product(R2)
  res = R.scalar(RR)
  print("d =", d, "n =", n, "k =", k, "dim =", res)
  return res
```

Listing 1 – Calculation of Kronecker coefficients using Sage software

```
def generate_lattice_words(word, cnt, lambda):
    if len(word) == sum(lambda):
        result.apend(word)
        return

    for i = 1 ... len(lambda):
        if cnt[i] < lambda[i] and (i == 1 or cnt[i-1] > cnt[i]):
            word.append(i)
            cnt[i] += 1
            generate_lattice_words(word, cnt, lambda)
            cnt[i] -= 1
            word.pop()
```

Listing 2 – Python-like realization of the Algorithm A' for lattice word search

```cpp
/*  A point is the d-tuple of integers.
    Each table is represented as the vector of points,
    with each point being a column. */
void gen_magic_cubes_rec(
    int n,  // the number of points in slice
    int k, // the size of a cube
    int id, // current point id
    vector<point> &cur, // vector of taken points
    vector< vector<int> > &cnt, // slice marginals
    vector< vector<point> > &all, // all tables
    bool no_rep, // true if repititions are not allowed
    ) {
    if (sz(cur) == n * k) {
        all.pb(cur);
        return;
    }
    if (id == sz(CUBE))
        return;
    bool skip = false;
    /* ensure the number of elements in the slice */
    for (int i = 0; i < d; i++) {
        if (cnt[i][CUBE[id][i]] == n) {
            skip = true;
            break;
        }
    }
    /* Generate only rows corrsponding to lattice words */
    for (int i = 0; i < d; i++) {
        int xi = CUBE[id][i];
        if (xi > 0 && cnt[i][xi-1] == cnt[i][xi]) {
            skip = true;
            break;
        }
    }
    if (!skip) {
        cur.pb(CUBE[id]);
        for (int i = 0; i < d; i++)
            cnt[i][CUBE[id][i]]++;

        gen_magic_cubes_rec(n,k,id+no_rep,cur,cnt,all,no_rep);

        cur.pop_back();
        for (int i = 0; i < d; i++)
            cnt[i][CUBE[id][i]]--;
    }
    gen_magic_cubes_rec(n,k,id+1,cur,cnt,all,no_rep);
}
```

Listing 3 – C++ realization of Algorithm A for generating sets $A^+(n,k)$ or $B^+(n,k)$ tables.

```cpp
vector<permutation> get_GST(word s, word t) {
    vector<permutation> Ys = get_young_subgroup(s); // stabilizer of a word s
    vector<permutation> Yt = get_young_subgroup(t); // stabilizer of a word t
    permutation h = get_nonzero_shift(s, t); // ensure that sgn_s(pi * t) != 0
    vector<permutaiton> res = Ys * {h} * Yt; // set product
    return res;
}
int get_ats(table T, table S) {
    int stab = get_stab(S); // stabilizer of table S
    /*
        double coset of form Y_{T[0]} omega Y_{S[0]}
        where T[0] and S[0] are the first rows of tables
    */
    vector<vi> doubleCoset = get_GST(T[0], S[0]);
    int ats = 0;
    for (permutation p in GST) {
        int x = sgn(T, p * S);
        ats += x;
    }
    ats /= stab;
    return ats;
}
void polynomial_search(int d, int n, int k) {
    auto A = gen_magic_sets(d, n, k, true); // repretitions allowed
    auto B = gen_magic_sets(d, k, n, false); // repetitions not allowed
    auto M = Matrix(A.size(), B.size());
    for (table T : B)
        for (table S : A) {
            int ats = get_ats(T, S);
            M[T,S] = ats;
        }
    result = gauss_jordan_elimination(M)
    return result
}
```

Listing 4 – C++ implementation of Algorithm B for calculation of the coefficients $a(T, S)$.