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Report on the dissertation of Farukh Mashurov "Special Tortkara algebras and assosymmetric algebrasâ"

An important direction in the theory of algebras is the study of relations between properties of an algebra A and of its adjoined commutative algebra $A^{(+)}$ and anticommutative algebra $A^{(-)}$ obtained by introducing on the vector space of A the symmetric (Jordan) product $a \circ b = \frac{1}{2}(ab + ba)$ and the antisymmetric product (commutator) $[a, b] = ab - ba$.

The first approach in this study concerns with the identities of algebras A, $A^{(+)}$, $A^{(-)}$. Let $\mathcal{M}$ be a variety of algebras defined by some set of identities. Define $\mathcal{M}^{(+)}$ and $\mathcal{M}^{(-)}$ as the classes of all algebras of the types $A^{(+)}$ and $A^{(-)}$, respectively, for all $A \in \mathcal{M}$. Then the natural questions are:

- What identities are verified in the algebras from the classes $\mathcal{M}^{(+)}$ and $\mathcal{M}^{(-)}$?
- Do the classes $\mathcal{M}^{(+)}$, $\mathcal{M}^{(-)}$ form varieties?
- What identities define the varieties $\overline{\mathcal{M}^{(+)}}$, $\overline{\mathcal{M}^{(-)}}$ generated by the classes $\mathcal{M}^{(+)}$, $\mathcal{M}^{(-)}$?

A classical example of such an approach is the variety of associative algebras $\mathcal{A}ss$. The anticommutative and commutative algebras $A^{(-)}$ and $A^{(+)}$ of an associative algebra A satisfy Jacobi and Jordan identities, respectively, that is,

$$\mathcal{A}ss^{(-)} \subseteq Lie, \ \mathcal{A}ss^{(+)} \subseteq Jord,$$

where Lie and $Jord$ are the varieties of the Lie and the Jordan algebras, respectively. By the famous Poincare-Birkhoff-Witt (PBW) theorem, any Lie algebra L can be imbedded in the algebra $A^{(-)}$ for an appropriate associative algebra A. In other words,

$$\mathcal{A}ss^{(-)} = \overline{\mathcal{A}ss^{(-)}} = Lie.$$

In the case of Jordan product, we have a quite different situation. A.Albert and L.Paige proved that $\overline{\mathcal{A}ss^{(+)}} \subsetneq Jord$ and P. Cohn showed that $\mathcal{A}ss^{(+)} \subsetneq \overline{\mathcal{A}ss^{(+)}}$, hence we have the strong inclusions

$$\mathcal{A}ss^{(+)} \subsetneq \overline{\mathcal{A}ss^{(+)}} \subsetneq Jord.$$

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The algebras from the class $\mathcal{A}ss^{(+)}$ are called *special Jordan algebras*, and the algebras from the variety $\overline{\mathcal{A}ss^{(+)}}$ are called *i*-special. The free algebra in $\overline{\mathcal{A}ss^{(+)}}$ is called *the free special algebra*. A.I.Shirshov proved that the free 2-generated Jordan algebra is special, and later P.Cohn proved that its homomorphic images are special as well, hence any 2-generated Jordan algebra is special. In the case of 3 generators, P.Cohn and A.Albert with L.Paige constructed examples of *i*-special nonspecial and non-*i*-special Jordan algebras, respectively.

Another important question in this line of research is the search for criteria to determine whether an element of free algebra is a Lie or a Jordan element. Let $A(X)$ be a free algebra on a set X of $\mathcal{M}$. An element of the algebra $A(X)$ is called a *Lie element* if it can be expressed by elements of X in terms of commutators. Similarly, an element of $A(X)$ is called a *Jordan element* if it can be expressed by elements of X in terms of Jordan products There are two well-known Lie criteria for free associative algebras: the Specht-Wever-Dynkin criterion and the Friedrich criterion. Jordan elements in a free associative algebra were described by P. Cohn only for the set of generators containing no more than three elements. He showed that an element is Jordan if and only if it is symmetric under the reverse involution. Using this criterion, some structural results concerning the theory of Jordan algebras were obtained.

A. Dzhumadildaev, M.Bremner, and P.Kolesnikov studied the properties of adjoint commutative and anticommutative algebras for the variety *Zinb* of so called *Zinbiel algebras*. More exactly, they proved that

$$Zinb^{(-)} \subsetneq \overline{Zinb^{(-)}} \subsetneq Tortk,$$

where *Tortk* is the class of so called *Tortkara algebras*. On the other hand, P. Kolesnikov proved that $Zinb^{(+)} \subsetneq \overline{Zinb^{(+)}}$.

In the first part of the dissertation the author continues the study of relations between classes of Zinbiel and Tortkara algebras. Here the following results are obtained:

- The criteria for determining Lie and Jordan elements in a free Zinbiel algebra is obtained;
- A basis for a free special Tortkara algebra is constructed;
- An exceptional homomorphic image of a free special Tortkara algebra with three generators is constructed;
- An analogue of Shirshov-Cohn theorem for Tortkara algebras is proved. That is, the speciality of any Tortkara algebra with two generators is proved.

The second part of the dissertation is devoted to the investigation of assosymmetric algebras in terms of their associated commutator Lie algebras. Lie ideals of free assosymmetric algebras of finite classes were studied. More exactly, the following results were obtained.

- If A is an assosymmetric algebra of finite class, then the commutator ideal $([A, A])$ is nilpotent of nilpotent index less or equal to the class of A;

- It was proved that $A_{[i]}A_{[j]} \subseteq A_{[i+j-1]}$ if i or j is odd for every assosymmetric algebra A where $A_{[i]}$ is the i-th commutator ideal of A.

Finally, in the last part of dissertation the author classified nilpotent assosymmetric algebras of small dimensions.

All the results of the dissertation are published in reviewed journals. They are new and interesting. The author demonstrated a good knowledge of the theory of non-associative algebras and the ability to apply the methods of this theory to investigation of new classes of algebras.

In my opinion, the dissertation meets the requirements for PhD dissertations, and its author Farukh Mashurov deserves to be awarded the PhD degree.

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