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Special Tortkara algebras and assosymmetric algebras

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CONTENT

NORMATIVE REFERENCES

This dissertation uses references to the following standards:

- SN RK 4.04-04-2013, Street lighting of urban communities and rural settlements.
- Rules for awarding academic degrees dated March 31, 2011 No.127
- GOST 7.32-2001 (changes from 2006). Research report. Structure and rules of registration.
- GOST 7.1-2003. Bibliographic record. Bibliographic description. General requirements and rules for compilation.

NOTATIONS

The following notations are used in this dissertation:

$\mathcal{M}$	a variety of algebras defined by some set of identities
$\mathcal{M}^{(+)}$	a class of all algebras of the types $A^{(+)}$ defined by $\{x,y\} = xy + yx$
$\mathcal{M}^{(-)}$	a class of all algebras of the types $A^{(-)}$ defined by $[x,y] = xy - yx$
$A(X)$	a free nonassociative algebra generated by set X
$Zin(X)$	a free Zinbiel algebra generated by set X
$T(X)$	a free Tortkara algebra generated by set X
$ST(X)$	a free special Tortkara algebra generated by set X
T_n	n-th homogenous part of $T(X)$
$a \sqcup\!\sqcup b$	the shuffle product of a and b
x_α	is $x_{i_1} \cdots x_{i_{n-1}} x_{i_n}$ for $\alpha = i_1 \cdots i_{n-1}\ i_n$
$\overline{x_\alpha}$	is a skew-right-commutative element of the form $x_{i_1} \cdots x_{i_{n-1}} x_{i_n} - x_{i_1} \cdots x_{i_{n-2}} x_{i_n} x_{i_{n-1}}$ for $\alpha = i_1 \cdots i_{n-1}\ i_n$
$\mathrm{Id}(A_{[i]})$	the i-th commutator ideal of A
AB	is spanned by $\{ab \mid a \in A, b \in B\}$
$[A,B]$	is spanned by $\{[a,b] \mid a \in A, b \in B\}$
$A \circ B$	the ideal of A generated by $[A,B]$.
$Z^2(A,V)$	the space of all cocycles $\theta : A \times A \longrightarrow V$
$B^2(A,V)$	the space of all coboundaries $\theta : A \times A \longrightarrow V$
$H^2(A,V)$	the second cohomology space defined as $Z^2(A,V)/B^2(A,V)$
$\mathrm{A}nn(\theta)$	the annihilator of θ
$\mathrm{A}ut(A)$	the automorphism group of the algebra A
A_j^{i*}	*j*-th *i*-dimensional nilpotent non-pure assosymmetric algebra A with identity $xyz = 0$
A_j^i	*j*-th *i*-dimensional nilpotent pure assosymmetric algebra A without identity $xyz = 0$
$\mathfrak{N}_i$	*i*-dimensional algebra with zero product
$(A)_{i,j}$	*j*-th *i*-dimensional central extension of A

INTRODUCTION

The presented dissertation is devoted to the special Tortkara algebras and nilpotency of assosymmetric algebras associated with Lie ideals and their algebraic classification in low dimensions.

The class of Tortkara algebras is a new class of nonassociative algebras discovered by A. Dzhumadil'daev [1]. Nonassociative algebras play an important role in many areas of mathematics. It is known that nonassociative algebras, such as Lie and Jordan algebras, arose within the framework of physics and have been extensively developed due to their applications in this field of science. Therefore, the classical theory of nonassociative algebras is mainly based on the study of Lie and Jordan algebras.

Let $\mathcal{M}$ be a variety of algebras defined by some set of identities. We define $\mathcal{M}^{(+)}$ and $\mathcal{M}^{(-)}$ as the classes of all algebras of the types $A^{(+)}$ and $A^{(-)}$ defined by the anticommutator $\{x, y\} = xy + yx$ and the commutator $[x, y] = xy - yx,$ respectively on the same vector space of $A \in \mathcal{M}$ for all $x, y \in A$. These two products usually connect two different known varieties of algebras, sometimes leading to new interesting classes of algebras. A classic example of such an approach is the variety of associative algebras. Recently, there has been a wide interest in studying other new types of nonassociative algebras over the commutator and anticommutator, such as Novikov, assosymmetric, bicommutative, Leibniz, and other algebras [2 – 6].

The commutator and anticommutator algebras of an associative algebra are known to satisfy Jacobi and Jordan identities, respectively. According to the well-known Poincare-Birkhoff-Witt (PBW) theorem, there are two independent identities, namely anticommutativity and Jacobi identities, which provide a complete list of identities for the commutator algebra of an associative algebra. This means that every identity satisfied by the commutator product in every associative algebra is a consequence of anticommutativity and Jacobi identities. However, the situation is different in the case of the anticommutator, as there is no embedding theorem. P. Cohn in [7] showed that a free special Jordan algebra with three generators has an exceptional homomorphic image. Consequently, there is no analogue of the PBW theorem for Jordan algebras. Also, it is known that the Glennie identity of degree eight exists, which is not a consequence of commutativity and Jordan identities. In this situation, many different interesting questions arise: studying the speciality of Jordan algebras, finding special identities, and others. In 1956 Shirshov proved that every Jordan algebra with two generators is special. This result gave further development of the theory of Jordan algebras. In this dissertation, we prove analogies of Cohn's and Shirshov's theorems for the free special Tortkara algebras.

Another important question in this line of research is the search for criteria to determine whether an element of free algebra is a Lie or a Jordan element. Let $A(X)$ be a free algebra on a set X of $\mathcal{M}$. An element of the algebra $A(X)$ is called a Lie element if it can be expressed by elements of X in terms of commutators. Similarly, an element of $A(X)$ is called a Jordan element if it can be expressed by elements of X in terms of anticommutators. There are two well-known Lie criteria for free associative algebras: the Specht-Wever-Dynkin criterion [8, 9] and the Friedrich criterion [10].

Jordan elements in free associative algebra were described by P. Cohn [7, p. 259] only for the set of generators containing no more than three elements. He showed that an element is Jordan if and only if it is symmetric under the involution map. Using this criterion, some structural results concerning the theory of Jordan algebras were obtained. Based on Cohn's result D. Robbins developed the study of Jordan elements in the free associative algebras [11]. But here, we give both Lie and Jordan criteria for elements in a free Zinbiel algebra.

A. Dzhumadil'daev proved that Zinbiel algebra under commutator satisfies the anticommutativity and Tortkara identities [1, p. 3911]. M. Bremner in [12], using representation theory, studied special identities in terms of the triple product of Tortkara defined as $[a, b, c] = [[a, b], c]$ in a free Zinbiel algebra and discovered identities in degrees five and seven in terms of the triple product. Recently, some geometric interpretations of Tortkara algebras have emerged in data science [13, 14]. The algebraic and geometric classifications of 5- and 6-dimensional Tortkara algebras were obtained in [15 – 17]. P. Kolesnikov in [18] shows that the class of all special Tortkara algebras does not form a variety. In the anticommutator case, he showed that there exists a homomorphic image of a free anticommutator algebra from a single generator, which is not embedded in the anticommutator algebra of the Zinbiel algebra. In addition, he asked a question about the maximum number of free generators for which all homomorphic images of a free special Tortkara algebra are special. The first part of the dissertation is devoted to the study of the above questions for Zinbiel and Tortkara algebras.

The second approach in our investigation is determining the structure of a Lie-admissible algebra when its related Lie algebra satisfies certain properties. Many of the properties of commutator subgroups had analogues in the theory of associative algebras in [19], with a suitable definition of "commutator ideals". Jennings in [19, p. 341] extended the concepts of a "nilpotent group" and a "solvable group" to a ring. He proved that if A is an associative algebra over a field with characteristics not equal to 2, if the associated Lie algebra is solvable, then A is solvable. Moreover, he obtained that if A is an associative algebra whose associated Lie algebra is nilpotent, then the ideal $A \circ A$ of A is generated by the set $\{ab - ba | a, b \in A\}$ is nilpotent. In [20] established that if A is an associative algebra over a field $\mathbb{F}$ whose associated Lie algebra is solvable, and if the characteristic of $\mathbb{F}$ is neither 2 nor 3, then $A \circ A$ is nil. If the associated Lie algebra of the associative algebra A over a field of characteristic $p > 0$ is either nilpotent or solvable with $p > 2$, then the ideal $A \circ A$ is nil of bounded index [21]. Assosymmetric algebras are introduced by Kleinfeld which come close to being associative [22]. Assosymmetric algebras as associative algebras under commutator are Lie-admissible algebras [23]. Kleinfeld proved that an assosymmetric ring of characteristic different from 2 and 3, without ideals $I \neq 0$, such that $I^2 = 0$ is associative. In addition, assosymmetric algebras were studied in [24 – 26]. The basis of free assosymmetric algebras was presented in [27]. Pokrass and Rodabaugh proved that each solvable assosymmetric ring of characteristics different from 2 and 3 is nilpotent [28]. We are continuing the investigation of Lie-admissible algebras such as assosymmetric algebras in terms of their associated Lie algebras.

The third considered problem is the classical problem in nonassociative algebra theory is to classify (up to isomorphism) the algebras of dimension n arising from a given variety described by a set of polynomial identities. It is common to concentrate on small dimensions, and there are two basic classification approaches: algebraic and geometric. These two methodologies have been used to study associative, Jordan, Lie, Leibniz, Zinbiel, and others, see [29 – 35] and references therein. We focus on the classification of the finite-dimensional nilpotent assosymmetric algebras. The key step of the method of algebraic classification of nilpotent assosymmetric algebras is the calculation of central extensions of small dimensional algebras. Firstly, Skjelbred and Sund devised a method for classifying nilpotent Lie algebras employing central extensions [36]. Moreover, the method was used to describe different varieties of nilpotent algebras of small dimensions such as the 4-dimensional nilpotent: associative algebras, Novikov algebras, bicommutative algebras, and Zinbiel algebras [32, p. 4, 37 – 39], all the 5-dimensional nilpotent Jordan algebras [34, p. 216], all the 6-dimensional nilpotent Lie algebras [33, p. 646], all the 6-dimensional nilpotent Malcev algebras [40] and some others.

The goal of the research. The goal of this research is to continue the study of Zinbiel algebras and assosymmetric algebras with respect to commutators. Specifically, the research aims to study homomorphic images of free special Tortkara algebras using Lie elements in the free Zinbiel algebra and provide an answer to a question previously posed in [18, p. 70]. An analogy of the classical Cohn's theorem in Jordan algebras for free special Tortkara algebras is also obtained. Furthermore, the research aims to generalize some properties of associative algebras to assosymmetric algebras related to Lie ideals. The final part of the research is devoted to the algebraic classification of nilpotent assosymmetric algebras, developing a unified algorithm using Wolfram Mathematica code to reduce the computational parts of the classification problem for finite dimensional nilpotent algebras and demonstrate it with a new classification of nilpotent assosymmetric algebras [41 – 43].

General methodology of the research. We use methods of structural and combinatorial theory of free Zinbiel and assosymmetric algebras. We study the basic methods of constructing central extensions of nonassociative algebras. We obtain the algebraic classification of small dimensional nilpotent assosymmetric algebras by the Skjelbred-Sund classification method [43, p. 154].

Scientific novelty. The main results of the first part of the dissertation are as follows:

- The criteria for determining Lie and Jordan elements in a free Zinbiel algebra is obtained;
- A basis for a free special Tortkara algebra is described;
- An exceptional homomorphic image of a free special Tortkara algebra with three generators is constructed;
- An analogue of Cohn's theorem for a free special Tortkara algebra is proved. That is, the speciality of any homomorphic image of a free special Tortkara algebra with two generators is proved;
- It was proved that there is no special identity with two generators.

For every assosymmetric algebra A we form a series of ideals

$$H_1 := A,\ H_{i+1} := H_i \circ A \quad for \quad i \geq 1.$$

It is said that A is of finite class if $H_n = (0)$ for some positive integer n. For the minimal integer n such that $H_n = (0)$, we call $n - 1$ the class of A [19, p. 343].

The main results of the second part of the dissertation are as follows:

- It was obtained that if A be an assosymmetric algebra of finite class, then $A \circ A$ is nilpotent of nilpotent index less or equal to the class of A;
- It was proved that $Id(A_{[i]})Id(A_{[j]}) \subseteq Id(A_{[i+j-1]})$ if i or j is odd for every assosymmetric algebra A, where $Id(A_{[i]})$ is the commutator ideal of A;
- The algebraic classification of nilpotent 4-dimensional assosymmetric algebras is obtained;
- The algebraic classification of nilpotent 5- and 6-dimensional assosymmetric algebras with one generator is obtained.

Theoretical and practical significance. The theoretical significance of this work lies in advancing the understanding of the structures of algebras and PI-theory. The results obtained in this research can be used to develop the theory of these structures further and to gain a deeper understanding of how they behave. Additionally, the findings can be applied to the study of finite dimensional assosymmetric algebras and free Tortkara and Zinbiel algebras.

In terms of practical significance, the results of this research can be used in various fields that rely on the use of algebras, such as mathematics, physics, and computer science. The results can also be used to improve the methods used to classify algebras and to develop new algorithms for solving problems related to algebras.

Publications. During the period of doctoral studies, 7 publications were published in international journals. Scopus and Thomson Reuters index these journals. The main results on the topic of the dissertation were published in the form of articles in peer-reviewed journals [41 – 44]. There are 3 articles that are not related to the topic of the dissertation [45 – 47]. Moreover, the authors of the published work [43] were awarded the Leader of Science Web of Science Award 2020 in the category of the most cited author from Kazakhstan by Clarivate Analytics.

The results of this dissertation were reported at:

- “Annual Scientific April Conference”, Institute of Mathematics and Mathematical Modeling (2022, Almaty, Kazakhstan);
- the scientific seminar of the Institute of Mathematics named after V.I. Romanovsky (2021, Tashkent, Uzbekistan);
- the scientific seminar of Astana IT University (2021, Nur-Sultan, Kazakhstan).
- III International Workshop on “Non-Associative Algebras in Malaga”, University of Malaga (2020, Malaga, Spain);
- the regular scientific seminar of the School of Mathematics and Cybernetics of the Kazakh-British Technical University (2019-2021, Almaty, Kazakhstan);
- the algebraic seminar of the Faculty of Engineering and Natural Sciences of the Suleyman Demirel University (2019-2021, Kaskelen, Kazakhstan);

The structure and scope of the thesis.

The dissertation consists of an introduction, three chapters, a conclusion, a list of references, and an appendix, for a total of 86 pages.

In Chapter 1, the fundamental notions and properties of nonassociative algebras are defined and recalled. Additionally, known results about specific types of nonassociative algebras, such as Jordan, Zinbiel, Tortkara, and assosymmetric algebras, are presented.

The next chapter is devoted to the study of free Zinbiel algebras over a commutator. The first section of this chapter is dedicated to obtaining the main lemmas, which are subsequently used to prove the main theorems of the chapter.

Let $X = \{x_1, x_2, \dots\}$ and $Zin(X)$ be a free Zinbiel algebra on X. Define a linear map $p: Zin(X) \to Zin(X)$ on base elements as follows

$$p(x_i) = -x_i,$$

$$p(x_i x_j) = x_j x_i,$$

$$p\left(x_{i_1} x_{i_2} \cdots x_{i_m} yz\right) = x_{i_1} x_{i_2} \cdots x_{i_m} zy, \qquad m \geq 1$$

where $y, z \in X$.

The first main result of this chapter is the following theorem, which gives us the Lie criterion for elements in a free Zinbiel algebra:

Theorem 2.2.8 *Let f be a Zinbiel element of $Zin(X)$. Then f is a Lie element if and only if $p(f) = -f$.*

The next theorem demonstrates a base of free special Tortkara algebra:

Theorem 2.2.9 *The set of skew-right-commutative elements $\overline{x_\alpha}$, where $\alpha \in \Gamma$, forms base of $ST(X)$.*

Moreover, we show that every identity with two generators is a consequence of anticommutativity and Tortkara identities:

Theorem 2.4.2 *The free Tortkara algebra $T(\{x, y\})$ is special.*

The next theorem is an analogue of Cohn's theorem on the speciality of homomorphic images of the free special Jordan algebras with two generators [7, p. 261]. For a free special Tortkara algebra with three generators, we have an exceptional homomorphic image, we show this by constructing a counter-example.

Theorem 2.5.1 *Any homomorphic image of a free special Tortkara algebra with two generators is special. For the three generators case, this statement is not true: a homomorphic image of special Tortkara algebra with three generators might be non-special.*

The results of Chapter 2 were published in [44].

The third chapter of this dissertation focuses on the study of assosymmetric algebras of finite class and commutator ideals of assosymmetric algebras. The main objective of the initial section is to examine the properties of assosymmetric algebras of finite class and demonstrate that they possess similar characteristics to associative algebras of finite class. The results obtained in this section can be used to further develop the theory of assosymmetric algebras and gain a deeper understanding of how they behave. The last part of this chapter is devoted to the algebraic classification of nilpotent assosymmetric algebras, where a unified algorithm is developed using Wolfram Mathematica code to reduce the computational parts of the classification problem.

We obtain an analogue of Jennings' result from [19, p. 346] for assosymmetric algebras:

Theorem 3.1.6 *Let A be an assosymmetric algebra of finite class. Then $A \circ A$ is nilpotent of nilpotent index less than or equal to the class of A.*

We have a generalization of of Corollary 1.4 in [48] for associative algebras.

Theorem 3.1.8 *Let A be an assosymmetric algebra. Then we have the following $Id(A_{[i]})Id(A_{[j]}) \subseteq Id(A_{[i+j-1]})$ if i or j is odd.*

The results of this section were published in [49].

The final section of this chapter concentrates on the algebraic classification of finite dimensional nilpotent assosymmetric algebras. The section starts with an overview of the necessary background information to apply the well-known Skjelbred-Sund classification method and the algorithms we follow in writing the code. We provide new results to illustrate our unified symbolic computational approach.

Theorem 3.2.5 *Let A be a nonzero 4-dimensional complex nilpotent assosymmetric algebra. Then, A is isomorphic to one of the algebras listed in Table A.1 in Appendix A.*

Regarding the 5 and 6-dimensional nilpotent assosymmetric algebras, applying the same algorithm we have the following theorem:

Theorem 3.2.7 *Let A be a 5- or 6-dimensional complex one-generated nilpotent assosymmetric algebra, then A is isomorphic to an algebra from Table A.3 or Table A.5 in Appendix A.*

The results of this section were published in [41 – 43].

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1 THEORETICAL FRAMEWORK

This chapter presents the main notions, definitions, and theorems that we need in our theorems and in their proofs. In addition, in the last sections, we recall some known results in the theory of Jordan, Zinbiel, Tortkara and assosymmetric algebras. It is based on books [50 – 52].

1.1 Basic properties of algebras

Let A be a vector space over a field F with given bilinear mapping (usually called multiplication) $(x, y) \to x \circ y$ on $A \times A \to A$ such that

$$(x + y) \circ z = x \circ z + y \circ z,$$

$$x \circ (y + z) = x \circ y + x \circ z,$$

$$\lambda(x \circ y) = (\lambda x) \circ y = x \circ (\lambda y),$$

where for $\lambda \in F$ and for all $x, y, z \in A$. Then A is called an algebra over field F. The multiplication $x \circ y$ is often abbreviated by xy. The dimension of the algebra A is its dimension as a vector space. The algebra A is finite-dimensional if A is a finite-dimensional vector space.

Let X be a set. The free nonassociative algebra $F[X]$ over a field F from the set of generators X is defined by the following universal property: For any algebra A, any mapping $X \to A$ can be uniquely extended to the algebra homomorphism $F[X] \to A$. The cardinality of the set X is called the rank of $F[X]$. We can construct the free algebra $F[X]$, using a set $K[X]$ of nonassociative words of the set X which is defined inductively:

$$x \in K[X], \quad \forall x \in X,$$

$$x_1x_2, x_1(v), (w)x_2, \ (v)(w) \in K[\mathrm{X}]$$

for $x_1, x_2 \in X,\ v, w \in K[X]$. No other sequences of the elements from X and brackets are not contained in $K[X]$.

Let us define the multiplication on $K[X]$ as

$$x_1 \circ x_2 = x_1x_2, x_1 \circ w = x_1(w), v \circ x_2 = (v)x_2, w \circ v = (w)(v).$$

Now we consider $F[X]$ to be a set of formal sums

$$\{\sum_j \alpha_j w_j | \alpha_j \in F, w_j \in K[X]\}$$

and extend the operation of multiplication defined on $K[X]$ to $F[X]$ by the rule

$$(\sum_j \alpha_j w_j) \circ (\sum_i \beta_i v_i) = \sum_{ji} \alpha_j \beta_i (w_j \circ v_i),$$

where $\alpha_j, \beta_i \in F$ and $w_j, v_i \in K[X]$. We obtain the free nonassociative algebra $F[X]$ over a field F from the set of generators X. The elements of $F[X]$ are called *nonassociative polynomials* of (noncommutative) variables from the set X.

A *monomial* is a polynomial with only one term, which can be written in the form of αu, where α is a scalar from the field F, and u is a polynomial in the variables X. The *degree* of a monomial is the length of the word u. The degree of a polynomial is the highest degree of its monomials. A monomial αu is said to have a *multi-degree* $(n_1, \dots, n_k)$ if it contains x_i exactly n_i times, $n_k \neq 0, n_j \neq 0; j > k$. A *homogeneous polynomial* is a polynomial where all monomials have the same multi-degree.

A homogeneous polynomial is called *multilinear* if it is linear in any of its variables (it is homogeneous of multidegree $(1,1, \dots ,1)$).

The linearization of homogeneous polynomials is useful in the study of identities of algebras and in the study of varieties. The process of the linearization is described in detail e.g. in [51, p. 24].

Let A be an algebra over a field F with multiplication $\circ$. The multiplication $x \circ y$ is often abbreviated by xy. A nonassociative polynomial $f = f(x_1, x_2, \dots x_k)$ is called an *identity* of the algebra A if $f(a_1, a_2, \dots, a_k) = 0$ for any $a_i \in A$, where $i = 1,2, \dots, k$. We say that A satisfies the identity f or that the identity f is valid in A.

For example, an algebra A is *commutative* if it satisfies the identity $xy = yx$ for all $x, y \in A$. The algebra is called *noncommutative* if it is not commutative.

An *associative* algebra is an algebra with identity

$$(a, b, c) = (ab)c - a(bc) = 0.$$

An algebra is *nonassociative* if the above identity is not satisfied.

An algebra is *assosymmetric* algebra if it is defined by the following identities:

$$(x, y, z) = (x, z, y), \qquad (x, y, z) = (y, x, z),$$

where $(x, y, z) = (xy)z - x(yz)$.

A nonassociative algebra with identity

$$a(bc) = (ab)c + (ba)c \tag{1}$$

is called *(right)-Zinbiel* algebra. Such algebras are called dual of Leibniz or Zinbiel (read Leibniz in reverse order) algebras. Zinbiel algebras were introduced by J-L.Loday in [53].

An algebra A is *anticommutative* if it satisfies the identity

$$x^2 = 0 \tag{2}$$

for all $x \in A$. This implies that $xy = -yx$, and the converse holds in characteristic $\neq 2$.

The Jacobian in an anticommutative algebra is defined by

$$J(x, y, z) = (xy)z + (yz)x + (zx)y.$$

A *Lie* algebra L is an anticommutative algebra satisfying the Jacobi identity

$$J(x, y, z) = 0,$$

for all $x, y, z \in L$.

Anticommuative algebra with $Tortkara$ identity

$$(ab)(cb) = J(a, b, c)b \tag{3}$$

is called *Tortkara* algebra [1]. If the characteristic of the field is different from two, then the identity (3) has the following multilinear form

$$(ab)(cd) + (ad)(cb) = J(a, b, c)d + J(a, d, c)b. \tag{4}$$

An algebra J is called Jordan algebra, if it is a commutative algebra with the following identity

$$(x^2, y, x) = 0 \tag{5}$$

for all $x, y \in J$ [52].

A *subalgebra* B of an algebra A is a closed under multiplication: $BB \subset B$ (i.e. for any $a, b \in B$ the product ab belongs to B). A (two-sided) ideal I of an algebra A is a subalgebra closed under multiplication by A, i.e.

$$AI + IA \subseteq I.$$

Ideals 0 and A of the algebra A are called improper ideals. The theory of nonassociative algebras defined two subsets of A which do behave associatively: The nucleus $N(A)$ (or the associative center) of an algebra A is the set of elements $z \in A$ which associate with every pair of elements $a, b \in A$ in the sense that $(z, a, b) = (a, z, b) = (a, b, z) = 0$ That is

$$N(A) = \{z \in A | (z, A, A) = (A, z, A) = (A, A, z) = 0\}.$$

The center $Z(A)$ of an algebra A is the set of all elements $z \in A$ which commute and associate with all elements in A. That is

$$Z(A) = \{z \in N(A) | [z, A] = 0\}.$$

Note that $N(A)$ is an associative and $Z(A)$ is a commutative and associative subalgebra of A. Moreover $Z(A) \subseteq N(A)$.

A homomorphism of algebras $\phi: A \to A'$ is a homomorphism of vector spaces (i.e. a linear mapping) which saves multiplication,

$$\phi(ab) = \phi(a)\phi(b)$$

for each $a, b \in A$.

The set

$$Ker(\phi) = \{x \in A | \phi(x) = 0\}$$

is a *kernel* and a homomorphic *image* of A of the homomorphism ϕ is the set

$$Im(\phi) = \{\phi(x) | x \in A\}.$$

If ϕ is injective homomorphism then we say that an algebra A is embedded in A'. A homomorphism of algebras which is bijective is called isomorphism (of algebras). An endomorphism is a homomorphism of algebras $\phi: A \to A$. If I is an ideal of A then the mapping $A \to A/I$, such that $a \to a + I$, is called a natural (or canonical) homomorphism of algebras.

Theorem 1.1.1 (Fundamental theorem of homomorphism for algebras [50, p. 9]) *Let A, A' be algebras. Let I be an ideal of A, and $\phi: A \to A'$ be a homomorphism of algebras and $\tau: A \to A/I$ the natural homomorphism. Then there is a unique homomorphism $h: A/I \to A'$, such that $h(a + I) = h(a)$. Furthermore, h is an isomorphism if and only if ϕ is a surjective homomorphism and $Ker(\phi) = I$.*

Theorem 1.1.2 (Isomorphism theorem [50, p. 10]) (i) *If $\phi: A \to B$ is a homomorphism of algebras over the field F, then*

$$A/Ker(\phi) \cong Im(\phi) \subset B.$$

(ii) *If I_1 and I_2 are ideals of the algebra A with $I_1 \subset I_2$, then*

$$(A/I_2)/(I_1/I_2) \cong A/I_1.$$

(iii) *If S is a subalgebra of A and B is an ideal of A, then $B \cap S$ is an ideal of S and*

$$(B + S)/B \cong S/(B \cap S).$$

An algebra A is called *nilpotent* if $A^m = 0$ for some m, where A^i are defined by

$$A^1 = A,$$

$$A^{i+1} = A^iA + A^{i-1}A^2 + \dots + A^2A^{i-1} + AA^i.$$

An algebra A is called *solvable* if $A^{(m)} = 0$ for some m, where $A^{(i)}$ are defined by

$$A^{(0)} = A, A^{(i+1)} = A^{(i)}A^{(i)}, i > 0.$$

The minimal such m is called the *index of nilpotency (index of solvability*, respectively) of the algebra A. Clearly, any nilpotent algebra is solvable. The concepts of solvability and nilpotency are equivalent for associative algebras:

$$A^{(i+1)} = A^{(i)}A^{(i)} = A^{2^i}.$$

1.2 Variety of algebras

Let I be a set of polynomials from $F[X]$. Then the class of all algebras satisfying this set of identities I is called the *variety of algebras* over the field F defined by the set of identities I. A *subvariety* is a subset of a variety which is itself a variety. Algebras from the variety $\mathcal{M}$ are called shortly $\mathcal{M}$ - algebras. The variety consisting of only the zero algebra is called trivial. The variety is called *homogeneous* if, for every identity f satisfied in the variety $\mathcal{M}$, all the homogeneous components of f are also satisfied in $\mathcal{M}$.

Proposition 1.2.1 [51, p. 17] *Every variety of algebras over an infinite field is homogeneous.*

Proposition 1.2.2 ([51 p. 25]) *Over a field of characteristic zero any homogeneous identity is equivalent to a multilinear identity.*

Proposition 1.2.3 ([54], see also [55, p. 181]) *Over a field of characteristic zero any variety can be defined by multilinear identities.*

Example 1.2.4 *The variety of associative algebras $\mathcal{A}ssoc$ is defined by one identity*

$$f(x_1, x_2, x_3) = (x_1x_2)x_3 - x_1(x_2x_3).$$

Example 1.2.5 *The variety of assosymmetric algebras $\mathcal{A}ssosym$ is defined by the identities*

$$f_1(x_1, x_2, x_3) = (x_1, x_2, x_3) - (x_1, x_3, x_1),$$

$$f_2(x_1, x_2, x_3) = (x_1, x_2, x_3) - (x_2, x_1, x_3).$$

Example 1.2.6 *The variety of Zinbiel algebras $\mathcal{Z}inb$ is defined by the identity*

$$f(x_1, x_2, x_3) = x_1(x_2x_3) - (x_1x_2)x_3 - (x_2x_1)x_3.$$

Let us denote by

$$\mathcal{M}(S) = \{A \mid \text{an algebra A satisfies all the identities from } S\}$$

the variety of algebras defined by the set $S \subset F[X]$. Notice that $\mathcal{M}(S) = \mathcal{M}(id\langle S\rangle)$, where $id\langle S\rangle$ denotes the ideal generated by K. Similarly, denote

$$I(A) = \{f \in F[X] \mid f = 0 \text{ for all } a \in A\}$$

the set of all the identities that are satisfied in A, and

$$I(\mathcal{M}) = \cap \{I(A) \mid A \in \mathcal{M}\}.$$

Any variety of algebras is closed under homomorphisms, subalgebras, and direct products by results in [55, 56]. And to decide whether a class of algebras forms a variety is used following Birkhoff's (or HSP) theorem.

Theorem 1.2.7 (Birkhoff theorem [55, p. 172]) *A class of algebras M form a variety if and only if M is closed under Homomorphisms, Subalgebras and direct Products (HSP).*

The algebra $F[X]$ is called a $\mathcal{M}$-free (relatively free or free in the variety $\mathcal{M}$) with the set of generators X, if for any algebra $B \in \mathcal{M}$ every mapping

$$\varphi: X \to B \in \mathcal{M}$$

can be uniquely extended to a homomorphism of the algebras

$$\varphi_{\mathcal{M}}: F[X] \to B.$$

The $\mathcal{M}$-free algebra is not free in general but only in the variety $\mathcal{M}$, i.e. it satisfies identities (and their consequences) that define the variety $\mathcal{M}$. The construction of the $\mathcal{M}$-free algebra is explained by the following theorem:

Theorem 1.2.8 [51, p. 13] *Let $\mathcal{M}$ be a nontrivial variety with the system of defining identities I. Then for any set X the natural homomorphism $F[X] \to F[X]/I(V)$ is injective and the quotient algebra is free in the variety V with the free set of generators X. Any two free algebras in $\mathcal{M}$ with equivalent sets of free generators are isomorphic.*

The commutator in algebra A is the bilinear function

$$[x, y] = xy - yx.$$

The *minus algebra* $A^{(-)}$ of algebra A is the algebra with the same underlying vector space as A but with the multiplication $[x, y]$.

The Jordan product (or anticommutator) in algebra A is the bilinear function

$$\{x, y\} = xy + yx.$$

The *plus algebra* $A^{(+)}$ of algebra A over a field F is the algebra with the same underlying vector space as A but with $\{x, y\}$ as the multiplication.

1.3 Special Jordan algebras

The theory of associative algebras and Jordan algebras is particularly important in the study of algebra. Let $\mathcal{A}ssoc$ be a variety of associative algebras. Let $\mathcal{A}ssoc^{(+)}$ class of algebras of types $A^{(+)}$. Well known, that any algebra in $\mathcal{A}ssoc^{(+)}$ satisfies the Jordan identity (5). Jordan algebra J is *special* if it is isomorphic to a subalgebra of the algebra $A^{(+)}$ for some associative algebra A. Otherwise, it is *exceptional*.

The study of free algebras is very important. Here we shall need two of them: the *free (unital) associative algebra* $FA\{x_1, \dots, x_n\}$ and the *free (unital) special Jordan algebra* $SJ(\{x_1, \dots, x_n\})$, which is the Jordan subalgebra of $FA(\{x_1, \dots, x_n\})$ generated by $\{x_1, \dots, x_n\}$ and 1.

A fundamental result in the theory of free special Jordan algebras is Proposition 1.3.1, also known as the universal property of free special Jordan algebras.

Proposition 1.3.1 (The universal property of Free special Jordan algebras [51, p. 76]) *Let A be a special Jordan algebra with a unit* 1. *If* $y_1, \dots, y_n \in A$ *there is a unique homomorphism* $\varphi: SJ(\{x_1, \dots, x_n\}) \to A$ *such that* $\varphi(1) = 1$ *and* $\varphi(x_i) = y_i, for\ i = 1, \dots, n$.

Let $X = \{x_1, x_2, \dots\}$ be a set and $FA(X)$ be a free associative algebra generated by X. A polynomial in $FA(X)$ is called Jordan element of $FA(X)$ if it can be expressed by elements of X in terms of anticommutators. There is still no criterion that determines all Jordan elements in $FA(X)$. This problem is solved only for some subspaces of the space of all Jordan elements.

Lemma 1.3.2 (P. Cohn [7, p. 255]) *Let* α *be an ideal of free special Jordan algebra* $SJ(X)$ *and* $\{\alpha\}$ *is an ideal of free special associative algebras generated by the set* α. *Then* $SJ(X)/\alpha$ *is a special Jordan algebra if and only if* $\{\alpha\} \cap SJ(X) \subseteq \alpha$.

Let us define in $FA(X)$ the involution $rev(\cdot)$ by

$$rev(x_1 x_2 \dots x_n) = x_n \dots x_2 x_1,$$

and the element u in $FA(X)$ is called *reversible* if $rev(u) = u$. The $u = \{x_1 \dots x_n\} = x_1 \dots x_n + x_n \dots x_1$ is called a reversible element.

Theorem 1.3.3 (P. Cohn [7, p. 257]) *Every reversible element of* $FA(X)$ *can be expressed as a Jordan polynomial in generators* $x_1, x_2, \dots x_n$ *and the elements*

$$\{x_{i_1} x_{i_2} x_{i_3} x_{i_4}\}$$

where $i_1 < i_2 < i_3 < i_4$ and $i = 1,2, \dots$.

The expression $\{x_{i_1} x_{i_2} x_{i_3} x_{i_4}\}$ is called a *tetrad*.

Corollary 1.3.4 (P. Cohn [7, p. 259]) *If the number of generators is less than four then the free special Jordan algebra $SJ(\{x_1, x_2, \dots x_n\})$ coincide with the space of reversible elements in $FA(\{x_1, x_2, \dots x_n\})$.*

Theorem 1.3.5 (P. Cohn [7, p. 262]) *Let $SJ(\{x_1, x_2, x_3\})$ is free special Jordan algebra generated by x_1, x_2, x_3 and α be the ideal in $SJ(\{x_1, x_2, x_3\})$ generated by element $k = x^2 - y^2$. Then $SJ(\{x_1, x_2, x_3\})/\alpha$ is exceptional.*

1.4 Zinbiel algebras

In some papers Zinbiel algebras are called *dual Leibniz*, *chronological* or *pre-commutative algebras* [18, p. 145], [57 – 59].

Let $X = \{x_1, x_2, \dots\}$ be a set. Let $Zin(X)$ be the free Zinbiel algebra generated on X. For $a_1, \dots, a_n \in Zin(X)$ denote by $a_1 a_2 \cdots a_n$ a left-bracketed element $(\cdots (a_1 \circ a_2) \cdots) \circ a_n$. In [53, p. 190] it was proved that the following set of elements

$$V(X) = \cup_n \{x_{i_1} x_{i_2} \cdots x_{i_n} | x_1, \dots x_{i_n} \in X\}$$

forms a base of the free Zinbiel algebra $Zin(X)$.

Now we present some results from [60], about the Zinbiel algebras. Recall that an algebra A is Zinbiel, if for any $x, y, z \in A$ relations (1) are satisfied.

Theorem 1.4.1 [60, p. 197] *Let K be an algebraically closed field of characteristic $p \leq 0$. Then every finite-dimensional Zinbiel algebra is solvable.*

The next theorem gives information about solvable Zinbiel algebras.

Theorem 1.4.2 [60, p. 197] *Let K be a field of characteristic $p \leq 0$ and A be a solvable Zinbiel algebra with solvability length N. If $p = 0$ or $p > 2^N - 1$, then A is a nil-algebra with nil-index no greater than 2^N. Conversely, if A is a Zinbiel nil-algebra with nil-index N and if $p = 0$ or $p > N - 1$, then A is solvable with solvability length N.*

Theorem 1.4.3 [60, p. 197] *Let K be a field of characteristic $p \leq 0$. Every Zinbiel nil-algebra is nilpotent. If A is a nil-algebra with nil-index n, then the nilpotency index of A is no greater than $2^n - 1$.*

Corollary 1.4.4 [60, p. 197] *Every finite-dimensional, simple Zinbiel algebra over an algebraically closed field of characteristic $p \leq 0$ is isomorphic to the $1-dimensional$ algebra with trivial multiplication.*

Corollary 1.4.5 [60, p. 197] *Every finite-dimensional Zinbiel algebra over the field of complex numbers is nilpotent (and, hence solvable and nil). If $p > 0$, then every finite-dimensional Zinbiel algebra over an algebraically closed field of dimension less than $log_2(p + 1)$ and characteristic p is nilpotent (and hence solvable and nil).*

Let

$$Z(A) = \{z \in A | a \circ z = z \circ a = 0, \forall a \in A\}$$

be the center of A.

Corollary 1.4.6 [60, p. 197] *Let A be a finite-dimensional Zinbiel algebra over the field of complex numbers of dimension n. Then there exists $N < n$ such that the* *product of any* N *elements of* A *in any type of bracketing is equal to* 0. *Moreover,* A *has the nontrivial center* $Z(A) \neq 0$. *The same is true for any finite-dimensional Zinbiel algebra* A *over a field of characteristic* $p > 0$ *if* $n = dimA < log_2(p+1)$.

Let $Zinbiel$ be a variety of Zinbiel algebras. Define $Zinbiel^{(+)}$ and $Zinbiel^{(-)}$ as classes of algebras of types $A^{(+)}$ and $A^{(-)}$ defined on the space $A \in Zinbiel$ by *anti-commutator* $\{x,y\} = x \circ y + y \circ x$ and *commutator* $[x,y] = x \circ y - y \circ x$, respectively. Any algebra in $Zinbiel^{(+)}$ is commutative and associative [53, p. 191]. It was proved in [1] that any algebra in $Zinbiel^{(-)}$ satisfies the $Tortkara$ identity

$$[[a,b],[c,d]] + [[a,d],[c,b]] = [J(a,b,c),d] + [J(a,d,c),b]$$

where $J(a,b,c) = [[a,b],c] + [[b,c],a] + [[c,a],b]$ is the Jacobi of elements a,b,c. A *Tortkara* algebra is defined as anticommutative algebra that satisfies Tortkara identity.

Theorem 1.4.7 [1, p. 3911] *For any Zinbiel algebra* $(A,\circ)$, *its Lie algebra* $(A,[,])$, *where* $[a,b] = a \circ b - b \circ a$, *satisfies the identity Tortkara. Any identity of degree 3 of the category* $Zinbiel^{(-1)}$ *follows from the anticommutative identity. Any identity of degree 4 for the category* $Zinbiel^{(-1)}$ *follows from the identities anticommutativity and Tortkara.*

In [1] obtained that the algebra $A = (C[x],\star)$, where

$$a \star b = b \int_0^x (\int_0^x a\,dx)dx$$

is not Zinbiel algebra, but the corresponding algebra under a commutator satisfies the Tortkara identity.

Theorem 1.4.8 [1, p. 3911] *The algebra* $A = (C[x],\star)$ *satisfies the right-symmetry identity*

$$(x_1x_2)x_3 - (x_1x_3)x_2 = 0$$

and the identity of degree four

$$(x_1,x_2,[x_3,x_4]) + (x_1,x_3,[x_4,x_2]) + (x_1,x_4,[x_2,x_3]) = 0,$$

where $(x_1,x_2,x_3) = (x_1x_2)x_3 - x_1(x_2x_3)$. *Then, its minus algebra* $A^{(-1)}$ *satisfies the Tortkara identity.*

An identity is called *special* if it holds in any homomorphic image of a special Tortkara algebra but does not hold in all Tortkara algebras. We still do not know whether there exists a special identity.

P. Kolesnikov proved that the classes neither $Zinbiel^{(+)}$ nor $Zinbiel^{(-)}$ are closed under the operation of taking homomorphic images and are therefore not variety [18, p. 153-154]. In the commutator case, P.S Kolesnikov constructed an example of special Tortkara algebra on four generators which can not be embedded into the commutator algebra of a Zinbiel algebra.

Theorem 1.4.9 [18, p. 153] *The algebra* $Zin^{(-)}(\{x_1, x_2, x_3, x_4\})$ *has an exceptional homomorphic image.*

Corollary 1.4.10 [18, p. 153] *The class of all special Tortkara algebras is not variety.*

In the anti-commutator case, he showed that there is a homomorphic image of a free anti-commutator algebra on one generator that is not embedded into the anti-commutator algebra of a Zinbiel algebra.

Theorem 1.4.11 [18, p. 154] *The algebra* $Zin^{(+)}(\{x_1\})$ *has an exceptional homomorphic image.*

1.5 Assosymmetric algebras

In this section, we present some results about assosymmetric algebras. Assosymmetric algebras are a type of nonassociative algebra that are of interest in the study of identities of algebras and varieties.

Theorem 1.5.1 (E. Kleinfeld [22, p. 983]) *If* A *is an assosymmetric algebra without ideals* $I \neq 0$, *such that* $I^2 = 0$, *then* A *is associative, provided the characteristic of* A *is different from* 2 *and* 3.

Let $A(X)$ be a free assosymmetric algebra on X. Denote by $[x_{i_1} x_{i_2} \cdots x_{i_n}]$ or $x_{i_1} x_{i_2} \cdots x_{i_n}$ a *left-normed element* $(\cdots(x_{i_1} x_{i_2}) x_{i_3} \cdots) x_{i_n}$ for $x_{i_1}, \ldots, x_{i_n} \in X$. The element

$$\langle x_{i_1} \cdots x_{i_n}, y_{i_1} \cdots y_{i_m} \rangle =$$

$$x_{i_1}(x_{i_2}(\cdots(x_{i_n}[\ldots[(y_{i_1}, y_{i_2}, y_{i_3}), y_{i_4}] \cdots, y_{i_m}] \cdots))$$

is called *ordered expression*, where $x_{i_1}, \ldots x_{i_n}, y_{i_1}, \ldots, y_{i_m} \in X$ and we have order $x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_n}$ and $y_{i_1} \leq y_{i_2} \leq \cdots \leq y_{i_m}$. In addition, it is known that the set of left-normed and ordered expression elements forms a basis for the free assosymmetric algebra $A(X)$ over a field of characteristic $\neq 2,3$. This was shown in [27, p. 312], where a multiplication rule of the base elements was also given. These multiplication rules are summarized in Proposition 1.5.2, which includes equations (6) - (8) and (9).

Proposition 1.5.2 [27, p. 312]

$$\langle u_1, v_1 \rangle \langle u_1, v_1 \rangle = 0, \tag{6}$$

$$[x]\langle u, v \rangle = \langle xu, v \rangle, \tag{7}$$

$$\langle u, v \rangle [x] = \sum_{x = x_1 x_2} \langle x_1 u, x_2 v \rangle, \tag{8}$$

$$[x][y] = [xy] - \sum_{\substack{x=x_1x_2 \\ |x_2|\geq 1 \\ y=y_1y_2 \\ |y_2|\geq 2}} (|y_2 - 1|)\langle x_1y_1, x_2y_2\rangle. \tag{9}$$

Furthermore, in [28, p. 32], it was proved that each solvable assosymmetric algebra of characteristic different from 2 and 3 is nilpotent.

Theorem 1.5.3 (D. Pokrass and D. Rodabaugh [28, p. 32]) *Let A be a solvable assosymmetric ring of characteristic* $\neq 2{,}3$. *Then A is nilpotent.*

This result, stated in Theorem 1.5.3 by D. Pokrass and D. Rodabaugh, gives us the motivation to classify low-dimensional nilpotent assosymmetric algebras over a field of characteristic 0. To accomplish this task, we will first analyze the properties of homogeneous polynomials in assosymmetric algebras and employ the Skjelbred and Sund method to classify them.

2 SPECIAL TORTKARA ALGEBRAS

In this chapter, we consider Zinbiel algebras under commutators and anticommutators. We give criteria for Lie and Jordan elements in a free Zinbiel algebra and by using criteria we obtain the main results of this chapter. All results of this chapter is published in [44].

2.1 Definitions and notations

We recall definition of a linear map $p: Zin(X) \to Zin(X)$ on base elements as follows

$$p(x_i) = -x_i,$$

$$p(x_i x_j) = x_j x_i,$$

$$p\left(x_{i_1} x_{i_2} \cdots x_{i_m}\, yz\right) = x_{i_1} x_{i_2} \cdots x_{i_m} zy, \qquad m \geq 1$$

where $y, z \in X$.

For $a \in Zin(X) \backslash X$ set

$$\bar{a} \coloneqq a - p(a).$$

Since $p^2 = id$, it is clear that

$$p(\bar{a}) = -\bar{a}.$$

Let $n > 1$ be an integer and let Γ be set of sequences $\alpha = i_1 \cdots i_{n-1} i_n$ such that $i_{n-1} < i_n$. For $\alpha = i_1 \ldots i_{n-1} i_n \in \Gamma$ set

$$x_\alpha = x_{i_1} \cdots x_{i_{n-1}}\, x_{i_n}.$$

We call elements of the form $\overline{x_\alpha}$, where $\alpha \in \Gamma$, *skew-right-commutative* or shortly *skew-rcom* elements of $Zin(X)$.

For instance, for $x, y, z, t \in X$, we have

$$\overline{(xy)(zt)} = (xy)(zt) - p\left((xy)(zt)\right) =$$

$$\left((xy)z\right)t + \left(z\,(xy)\right)t - p\left(\left((xy)z\right)t + \left(z\,(xy)\right)t\right) =$$

$$xyzt + zxyt + xzyt - p\,(xyzt + zxyt + xzyt) =$$

$$xyzt + zxyt + xzyt - xytz - zxty - xzty =$$

$$\overline{xyzt} + \overline{zxyt} + \overline{xzyt},$$

$$\overline{x_{3124}} = x_3x_1x_2x_4 - x_3x_1x_4x_2.$$

Recall the definition of the Lie element. We say that for w in a free Zinbiel algebra $Zin(X)$ is *Lie element* if it can be shown as a linear combination of words on X under the product $[a, b] = ab - ba$. Similarly, an element $w \in Zin(X)$ is referred to as a *Jordan element* if it can be expressed as a linear combination of words on X using the product $\{a, b\} = ab + ba$. Next, we define $ST(X)$ as a free special Tortkara algebra on X under the commutator, i.e., subalgebra of $Zin(X)^{(-)} = (Zin(X), [\ ,\ ])$ generated by X. Furthermore, $J(X)$ is defined as a subalgebra of the $Zin(X)^{(+)} = (Zin(X), \{\ ,\ \})$ generated by X.

Define *Dynkin map* $D: Zin(X) \to Zin(X)$ on base elements as follows

$$D: x_{i_1}x_{i_2} \cdots x_{i_n} \mapsto \{\{\cdots\{x_{i_1}, x_{i_2}\}, \cdots\}, x_{i_n}\}.$$

2.2 Lie elements in a free Zinbiel algebra

In this section, we give Lie criterion for elements in a free Zinbiel algebra.

2.2.1 Shuffle permutations

Let $Sh_{m,n}$ be set of shuffle permutations, i.e.,

$$Sh_{m,n} = \{\sigma \in S_{n+m} | \sigma(1) < \cdots < \sigma(m), \sigma(m+1) < \cdots \sigma(m+n)\}.$$

For any positive of integers $i_1, \dots, i_m$ and $j_1, \dots, j_n$ denote by $Sh(i_1 \dots i_m; j_1 \dots j_n)$ set of sequences $\alpha = \alpha_1 \dots \alpha_{n+m}$ constructed by shuffle permutations $\sigma \in Sh_{m,n}$ by changing $\alpha_{\sigma(l)}$ to i_l if $l \leq m$ and to j_{l-m} if $m < l \leq m+n$.

For example,

$$Sh(12; 34) = \{1234,1324,3124,1342,3142,3412\},$$

$$Sh(23; 41) = \{2341,2431,4231,2413,4213,4123\}.$$

The following proposition, which was established in [53] for free left-Zinbiel algebras, can also be derived for free right-Zinbiel algebras..

Proposition 2.2.1. (Loday [53, p. 192])

$$\left(x_{i_1} \cdots x_{i_p}\right) \circ \left(x_{j_1} \cdots x_{i_q}\right) = \sum_{\sigma \in Sh\left(i_1 \dots i_p;\, j_1 \dots j_{q-1}\right)} x_{\sigma(1)} \cdots x_{\sigma(p+q-1)} x_{j_q}.$$

Proof. The validity of the formula can be established through induction on $n = p + q$ and the use of identity (1).

The shuffle product of two base elements $u = x_{i_1} \cdots x_{i_p}$ and $v = x_{i_{p+1}} \cdots x_{i_{p+q}}$ in the free Zinbiel algebra $Zin(X)$ is defined as follows:

$$u \sqcup\!\sqcup \ v = \sum_{\sigma \in Sh(i_1 \dots i_p; i_{p+1} \dots i_{p+q})} x_{\sigma(1)} \cdots \ x_{\sigma(p+q)}.$$

Proposition 2.2.2 *The shuffle product on $Zin(X)$ has the following properties:*

a) *the shuffle product is commutative and associative*

$$a \sqcup\!\sqcup \ b = b \sqcup\!\sqcup \ a, \ (a \sqcup\!\sqcup \ b) \sqcup\!\sqcup \ c = a \sqcup\!\sqcup \ (b \sqcup\!\sqcup \ c), \textit{for any } a, b, c \in Zin(X).$$

b) $\left(x_{i_1} \cdots x_{i_p}\right) \circ \left(x_{j_1} \cdots x_{j_q}\right) = \begin{cases} x_{i_1} \cdots x_{i_p} x_{j_1}, \ for\ q = 1 \\ (x_{i_1} \cdots x_{i_p} \sqcup\!\sqcup \ x_{j_1} \cdots x_{j_{q-1}}) x_{j_q}, \ for\ q > 1 \end{cases}$

c) $\left(x_{i_1} \cdots x_{i_p}\right) \sqcup\!\sqcup \left(x_{j_1} \cdots x_{j_q}\right) =$
$\begin{cases} x_{i_1} x_{j_1} + x_{j_1} x_{i_1}, \ for\ p = q = 1, \\ (x_{i_1} \sqcup\!\sqcup \ x_{j_1} \cdots x_{j_{q-1}}) \circ x_{j_q} + x_{j_1} \cdots x_{j_q} x_{i_1}, for\ p = 1, q > 1, \\ (x_{i_1} \cdots x_{i_{p-1}} \sqcup\!\sqcup \ x_{j_1} \cdots x_{j_q}) \circ x_{j_p} + (x_{i_1} \cdots x_{i_p} \sqcup\!\sqcup \ x_{j_1} \cdots x_{j_{q-1}}) \circ x_{j_q}, for\ p, q > 1 \end{cases}$

For example,

$$(ab) \circ \ (cd) = (abc + acb + cab)d = (ab \ \sqcup\!\sqcup \ c) \circ d,$$

$$(ab) \sqcup\!\sqcup (cd) = abcd + acbd + cabd + acdb + cadb + cdab =$$

$$(abc + acb + cab)d + (acd + cad + cda)b = (ab \sqcup\!\sqcup c)d + (a \sqcup\!\sqcup cd)b.$$

Proof. All these properties follow Proposition 2.2.1 and the definition of the shuffle product.

2.2.2 Products of skew-right-commutative elements

In the following lemma, we define the product of skew-right-commutative elements in the Zinbiel algebra.

Lemma 2.2.3 *Zinbiel product of skew-right-commutative elements can be presented as follows*

$$\overline{x_{i_1} \cdots x_{i_m}} \circ \overline{x_{j_1} x_{j_2}} = \overline{x_{i_1} \cdots x_{i_m} x_{j_1} x_{j_2}} - \overline{x_{i_1} \cdots x_{i_{m-2}} x_{i_m} x_{i_{m-1}} x_{j_1} x_{j_2}} +$$

$$\left(x_{i_1} \cdots x_{i_{m-1}} \ \sqcup\!\sqcup \ x_{j_1}\right) x_{i_m} x_{j_2} - \left(x_{i_1} \cdots x_{i_{m-2}} x_{i_m} \ \sqcup\!\sqcup \ x_{j_1}\right) x_{i_{m-1}} x_{j_2} -$$

$$\left(x_{i_1} \cdots x_{i_{m-1}} \ \sqcup\!\sqcup \ x_{j_2}\right) x_{i_m} x_{j_1} - \left(x_{i_1} \cdots x_{i_{m-2}} x_{i_m} \ \sqcup\!\sqcup \ x_{j_2}\right) x_{i_{m-1}} x_{j_1},$$

and for $m \geq 2,\ n \geq 3$

$$\overline{x_{i_1} \cdots x_{i_m}} \circ \overline{x_{j_1} \cdots x_{j_n}} =$$

$$\overline{(x_{i_1} \cdots x_{i_m} \text{ ш } x_{j_1} \cdots x_{j_{n-2}}) x_{j_{n-1}} x_{j_n}} - \overline{x_{i_1} \cdots x_{i_{m-2}} x_{i_m} x_{i_{m-1}} \text{ ш } x_{j_1} \cdots x_{j_{n-2}})\, x_{j_{n-1}} x_{j_n}} +$$

$$(x_{i_1} \cdots x_{i_{m-1}} \text{ ш } x_{j_1} \cdots x_{j_{n-1}}) x_{i_m} x_{j_n} - (x_{i_1} \cdots x_{i_{m-2}} x_{i_m} \text{ ш } x_{j_1} \cdots x_{j_{n-1}}) x_{i_{m-1}} x_{j_n} -$$

$$(x_{i_1} \cdots x_{i_{m-1}} \text{ ш } x_{j_1} \cdots x_{j_{n-2}} x_{j_n}) x_{i_m} x_{j_{n-1}} -$$

$$(x_{i_1} \cdots x_{i_{m-2}} x_{i_m} \text{ ш } x_{j_1} \cdots x_{j_{n-2}} x_{j_n}) x_{j_{m-1}} x_{j_{n-1}}.$$

Proof. Let us prove the first part of the lemma,

$$\overline{x_{i_1} \cdots x_{i_m}} \circ \overline{x_{j_1} x_{j_2}} = x_{i_1} \cdots x_{i_m} \circ\ x_{j_1} x_{j_2} - x_{i_1} \cdots x_{i_{m-2}} x_{i_m} x_{i_{m-1}} \circ x_{j_1} x_{j_2} -$$

$$x_{i_1} \cdots x_{i_m} \circ\ x_{j_2} x_{j_1} - x_{i_1} \cdots x_{i_{m-2}} x_{i_m} x_{i_{m-1}} \circ x_{j_2} x_{j_1} =$$

(by part **b** of Proposition 2.2.2)

$$(x_{i_1} \cdots x_{i_m} \text{ ш } x_{j_1}) x_{j_2} - (x_{i_1} \cdots x_{i_{m-2}} x_{i_m} x_{i_{m-1}} \text{ ш } x_{j_1}) x_{j_2} -$$

$$(x_{i_1} \cdots x_{i_m} \text{ ш } x_{j_2}) x_{j_1} + (x_{i_1} \cdots x_{i_{m-2}} x_{i_m} x_{i_{m-1}} \text{ ш } x_{j_2}) x_{j_1} =$$

(by the definitions of shuffle product and skew-rcom elements)

$$\overline{x_{i_1} \cdots x_{i_m} x_{j_1} x_{j_2}} - \overline{x_{i_1} \cdots x_{i_{m-2}} x_{i_m} x_{i_{m-1}} x_{j_1} x_{j_2}} +$$

$$\left(x_{i_1} \cdots x_{i_{m-1}} \text{ ш } x_{j_1}\right) x_{i_m} x_{j_2} - \left(x_{i_1} \cdots x_{i_{m-2}} x_{i_m} \text{ ш } x_{j_1}\right) x_{i_{m-1}} x_{j_2} -$$

$$\left(x_{i_1} \cdots x_{i_{m-1}} \text{ ш } x_{j_2}\right) x_{i_m} x_{j_1} - \left(x_{i_1} \cdots x_{i_{m-2}} x_{i_m} \text{ ш } x_{j_2}\right) x_{i_{m-1}} x_{j_1}.$$

Let $n \geq 3$,

$$\overline{x_{i_1} \cdots x_{i_m}} \circ \overline{x_{j_1} \cdots x_{j_n}} =$$

$$x_{i_1} \cdots x_{i_m} \circ x_{j_1} \cdots x_{j_n} - x_{i_1} \cdots x_{i_{m-2}} x_{i_m} x_{i_{m-1}} \circ x_{j_1} \cdots x_{j_n} -$$

$$x_{i_1} \cdots x_{i_m} \circ x_{j_1} \cdots x_{j_{n-2}} x_{j_n} x_{j_{n-1}} + x_{i_1} \cdots x_{i_{m-2}} x_{i_m} x_{i_{m-1}} \circ x_{j_1} \cdots x_{j_{n-2}} x_{j_n} x_{j_{n-1}} =$$

(by part **b** of Proposition 2.2.2)

$$(x_{i_1} \cdots x_{i_m} \text{ ш } x_{j_1} \cdots x_{j_{n-1}}) x_{j_n} - (x_{i_1} \cdots x_{i_{m-2}} x_{i_m} x_{i_{m-1}} \text{ ш } x_{j_1} \cdots x_{j_{n-1}}) x_{j_n} -$$

$$(x_{i_1}\cdots x_{i_m} \sqcup\!\sqcup\ x_{j_1}\cdots x_{j_{n-2}}x_{j_n})x_{j_{n-1}} + (x_{i_1}\cdots x_{i_{m-2}}x_{i_m}x_{i_{m-1}} \sqcup\!\sqcup\ x_{j_1}\cdots x_{j_{n-2}}x_{j_n})x_{j_{n-1}} =$$

(by part **c** of Proposition 2.2.2)

$$(x_{i_1}\cdots x_{i_{m-1}} \sqcup\!\sqcup\ x_{j_1}\cdots x_{j_{n-1}})x_{i_m}x_{j_n} - (x_{i_1}\cdots x_{i_m} \sqcup\!\sqcup\ x_{j_1}\cdots x_{j_{n-2}})x_{j_{n-1}}x_{j_n} -$$

$$(x_{i_1}\cdots x_{i_m} \sqcup\!\sqcup\ x_{j_1}\cdots x_{j_{n-1}})x_{i_{m-1}}x_{j_n} - (x_{i_1}\cdots x_{i_{m-2}}x_{i_m}x_{i_{m-1}} \sqcup\!\sqcup\ x_{j_1}\cdots x_{j_{n-2}})x_{j_{n-1}}x_{j_n} -$$

$$(x_{i_1}\cdots x_{i_{m-1}} \sqcup\!\sqcup\ x_{j_1}\cdots x_{j_{n-2}}x_{j_n})x_{i_m}x_{j_{n-1}} - (x_{i_1}\cdots x_{i_m} \sqcup\!\sqcup\ x_{j_1}\cdots x_{j_{n-2}})x_{j_n}x_{j_{n-1}} -$$

$$(x_{i_1}\cdots x_{i_{m-2}}x_{i_m} \sqcup\!\sqcup\ x_{j_1}\cdots x_{j_{n-2}}x_{j_n})x_{i_{m-1}}x_{j_{n-1}} -$$

$$(x_{i_1}\cdots x_{i_m}x_{i_{m-1}} \sqcup\!\sqcup\ x_{j_1}\cdots x_{j_{n-2}})x_{j_n}x_{j_{n-1}} =$$

(by definition of skew-right-commutative elements we obtain)

$$\overline{(x_{i_1}\cdots x_{i_m}\ \sqcup\!\sqcup\ x_{j_1}\cdots x_{j_{n-2}})x_{j_{n-1}}x_{j_n}} - \overline{x_{i_1}\cdots x_{i_{m-2}}x_{i_m}x_{i_{m-1}} \sqcup\!\sqcup\ x_{j_1}\cdots x_{j_{n-2}})\ x_{j_{n-1}}x_{j_n}} +$$

$$(x_{i_1}\cdots x_{i_{m-1}}\ \sqcup\!\sqcup\ x_{j_1}\cdots x_{j_{n-1}})x_{i_m}x_{j_n} - (x_{i_1}\cdots x_{i_{m-2}}x_{i_m}\ \sqcup\!\sqcup\ x_{j_1}\cdots x_{j_{n-1}})x_{j_{m-1}}x_{j_n} -$$

$$(x_{i_1}\cdots x_{i_{m-1}} \sqcup\!\sqcup\ x_{j_1}\cdots x_{i_{n-2}}x_{j_n})x_{i_m}x_{j_{n-1}} -$$

$$(x_{i_1}\cdots x_{i_{m-2}}x_{i_m} \sqcup\!\sqcup\ x_{j_1}\cdots x_{j_{n-2}}x_{j_n})x_{j_{m-1}}x_{j_{n-1}}.$$

This completes the proof.

In the next lemma we show that the commutator product of skew-rcom element by generator is a linear combination of skew-rcom elements.

Lemma 2.2.4

$$[\overline{x_{i_1}\cdots x_{i_{m-1}}x_{i_m}}, x_{j_1}] =$$

$$\begin{cases} \overline{x_{i_1}x_{j_1}}\ \ for\ m=1, \\ \overline{x_{i_1}x_{i_2}x_{j_1}} - \overline{x_{i_2}x_{i_1}x_{j_1}} - \overline{x_{j_1}x_{i_2}x_{i_1}}\ \ for\ m=2, \\ \overline{x_{i_1}\cdots x_{i_m}x_{j_1}} - \overline{x_{i_1}\cdots x_{i_{m-2}}x_{i_m}x_{i_{m-1}}x_{j_1}} - \overline{\left(x_{j_1} \sqcup\!\sqcup\ x_{i_1}\cdots x_{i_{m-2}}\right)x_{i_{m-1}}x_{i_m}}\ \ for\ m>2. \end{cases}$$

Proof. Since, $[x_{i_1}, x_{j_1}] = x_{i_1}x_{j_1} - x_{j_1}x_{i_1} = \overline{x_{i_1}x_{j_1}}$, we start proof of lemma from $m=2$,

$$[\overline{x_{i_1}x_{i_2}}, x_{j_1}] = \overline{x_{i_1}x_{i_2}} \circ x_{j_1} - x_{j_1} \circ \overline{x_{i_1}x_{i_2}} =$$

$$(x_{i_1}x_{i_2}) \circ x_{j_1} - (x_{i_2}x_{i_1}) \circ x_{j_1} - x_{j_1} \circ (x_{i_1}x_{i_2}) + x_{j_1} \circ (x_{i_2}x_{i_1}) =$$

$$\overline{x_{i_1}x_{i_2}x_{j_1}} - \overline{x_{i_2}x_{i_1}x_{j_1}} - \overline{x_{j_1}x_{i_2}x_{i_1}}.$$

Now suppose $m > 2$. Then

$$\left[\overline{x_{i_1}\cdots x_{i_{m-1}}x_{i_m}}, x_{j_1}\right] = \overline{x_{i_1}\cdots x_{i_{m-1}}x_{i_m}} \circ x_{j_1} - x_{j_1} \circ \overline{x_{i_1}\cdots x_{i_{m-1}}x_{i_m}} =$$

$$(x_{i_1}\cdots x_{i_{m-1}}x_{i_m}) \circ x_{j_1} - (x_{i_1}\cdots x_{i_m}x_{i_{m-1}}) \circ x_{j_1} -$$

$$x_{j_1} \circ (x_{i_1}\cdots x_{i_{m-1}}x_{i_m}) + x_{j_1} \circ (x_{i_1}\cdots x_{i_m}x_{i_{m-1}}) =$$

(by part **b** of Proposition 2.2.2)

$$x_{i_1}\cdots x_{i_{m-1}}x_{i_m}x_{j_1} - x_{i_1}\cdots x_{i_m}x_{i_{m-1}}x_{j_1} -$$

$$(x_{j_1} ш\ x_{i_1}\cdots x_{i_{m-1}})x_{i_m} + (x_{j_1} ш\ x_{i_1}\cdots x_{i_m})x_{i_{m-1}} =$$

(by part **c** of Proposition 2.2.2)

$$x_{i_1}\cdots x_{i_{m-1}}x_{i_m}x_{j_1} - x_{i_1}\cdots x_{i_m}x_{i_{m-1}}x_{j_1} -$$

$$x_{i_1}\cdots x_{i_{m-1}}x_{j_1}x_{i_m} + (x_{j_1} ш\ x_{i_1}\cdots x_{i_{m-2}})x_{i_{m-1}}x_{i_m} +$$

$$x_{i_1}\cdots x_{i_{m-2}}x_{i_m}x_{j_1}x_{i_{m-1}} + (x_{j_1} ш\ x_{i_1}\cdots x_{i_{m-2}})x_{i_m}x_{i_{m-1}} =$$

(by the definition of skew-right-commutative elements we obtain)

$$\overline{x_{i_1}\cdots x_{i_m}x_{j_1}} - \overline{x_{i_1}\cdots x_{i_{m-2}}x_{i_m}x_{i_{m-1}}x_{j_1}} - \overline{\left(x_{j_1} ш\, x_{i_1}\cdots x_{i_{m-2}}\right)x_{i_{m-1}}x_{i_m}}.$$

This completes the proof.

Lemma 2.2.5 *The commutator of skew-right-commutative elements is a linear combination of skew-right-commutative elements.*

Proof. If skew-right-commutative elements have degree 2, then a straightforward calculation shows that

$$\left[\overline{x_{i_1}x_{i_2}}, \overline{x_{j_1}x_{j_2}}\right] = \overline{\overline{x_{i_1}x_{i_2}}x_{j_1}x_{j_2}} - \overline{\overline{x_{j_1}x_{j_2}}x_{i_1}x_{i_2}} +$$

$$\overline{(x_{i_1} ш\ x_{j_1})x_{i_2}x_{j_2}} - \overline{(x_{i_1} ш\ x_{j_2})x_{i_2}x_{j_1}} + \overline{(x_{i_2} ш\ x_{j_2})x_{i_1}x_{j_1}} - \overline{(x_{i_2} ш\ x_{j_1})x_{i_1}x_{j_2}}.$$

The proof of the assertion is presented below under the assumption that the degree of skew-right commutative elements is at least 3. The case when one of the elements has degree 2 is established using similar way. So,

$$\left[\overline{x_{i_1}\cdots x_{i_m}}, \overline{x_{j_1}\cdots x_{j_n}}\right] = \overline{x_{i_1}\cdots x_{i_m}} \circ \overline{x_{j_1}\cdots x_{j_n}} - \overline{x_{j_1}\cdots x_{j_n}} \circ \overline{x_{i_1}\cdots x_{i_m}} =$$

(by Lemma 2.2.3)

$$\overline{(x_{i_1}\cdots x_{i_m} ш x_{j_1}\cdots x_{j_{n-2}})x_{j_{n-1}}x_{j_n}} - \overline{x_{i_1}\cdots x_{i_{m-2}}x_{i_m}x_{i_{m-1}} ш x_{j_1}\cdots x_{j_{n-2}})\, x_{j_{n-1}}x_{j_n}} +$$

$$(x_{i_1}\cdots x_{i_{m-1}} ш x_{j_1}\cdots x_{j_{n-1}})x_{i_m}x_{j_n} - (x_{i_1}\cdots x_{i_{m-2}}x_{i_m} ш x_{j_1}\cdots x_{j_{n-1}})x_{i_{m-1}}x_{j_n} -$$

$$(x_{i_1}\cdots x_{i_{m-1}} ш x_{j_1}\cdots x_{j_{n-2}}x_{j_n})x_{i_m}x_{j_{n-1}} -$$

$$(x_{i_1}\cdots x_{i_{m-2}}x_{i_m} ш x_{j_1}\cdots x_{j_{n-2}}x_{j_n})x_{j_{m-1}}x_{j_{n-1}} -$$

$$\overline{(x_{j_1}\cdots x_{j_n} ш x_{i_1}\cdots x_{i_{m-2}})x_{i_{m-1}}x_{i_n}} - \overline{x_{i_1}\cdots x_{j_{n-2}}x_{j_n}x_{j_{n-1}} ш x_{i_1}\cdots x_{i_{m-2}})\, x_{i_{m-1}}x_{i_m}} +$$

$$(x_{j_1}\cdots x_{j_{n-1}} ш x_{i_1}\cdots x_{i_{m-1}})x_{j_n}x_{i_m} - (x_{j_1}\cdots x_{j_{n-2}}x_{j_n} ш x_{i_1}\cdots x_{i_{m-1}})x_{j_{n-1}}x_{i_m} -$$

$$(x_{j_1}\cdots x_{j_{n-1}} ш x_{i_1}\cdots x_{i_{m-2}}x_{i_m})x_{j_n}x_{i_{m-1}} -$$

$$(x_{j_1}\cdots x_{j_{n-2}}x_{j_n} ш x_{i_1}\cdots x_{i_{m-2}}x_{i_m})x_{j_{n-1}}x_{i_{m-1}} =$$

(by part **a** of Proposition 2.2.2)

$$\overline{(x_{i_1}\cdots x_{i_m} ш x_{j_1}\cdots x_{j_{n-2}})x_{j_{n-1}}x_{j_n}} - \overline{x_{i_1}\cdots x_{i_{m-2}}x_{i_m}x_{i_{m-1}} ш x_{j_1}\cdots x_{j_{n-2}})\, x_{j_{n-1}}x_{j_n}} -$$

$$\overline{(x_{j_1}\cdots x_{j_n} ш x_{i_1}\cdots x_{i_{m-2}})x_{i_{m-1}}x_{i_n}} - \overline{x_{i_1}\cdots x_{j_{n-2}}x_{j_n}x_{j_{n-1}} ш x_{i_1}\cdots x_{i_{m-2}})\, x_{i_{m-1}}x_{i_m}} +$$

$$\overline{(x_{i_1}\cdots x_{i_{m-1}} ш x_{j_1}\cdots x_{j_{n-1}})x_{i_m}x_{j_n}} - \overline{(x_{i_1}\cdots x_{i_{m-2}}x_{i_m} ш x_{j_1}\cdots x_{j_{n-1}})x_{i_{m-1}}x_{j_n}} -$$

$$\overline{(x_{i_1}\cdots x_{i_{m-1}} ш x_{j_1}\cdots x_{j_{n-2}}x_{j_n})x_{i_m}x_{j_{n-1}}} -$$

$$\overline{(x_{i_1}\cdots x_{i_{m-2}}x_{i_m} ш x_{j_1}\cdots x_{j_{n-2}}x_{j_n})x_{j_{m-1}}x_{j_{n-1}}}.$$

This completes the proof.

Lemma 2.2.6 *If* u *is an element of* $ST(X)$, *then* $p(u) = -u$.

Proof. The proof is achieved by the fact that $ST(X)$ is generated by the commutator products on X and $[x, y] = \overline{xy}$ for any $x, y \in X$, and by using Lemma 2.2.4 and Lemma 2.2.5.

Now we prove that any skew-right-commutative element of $Zin(X)$ is a Lie element.

Lemma 2.2.7 *Let f is an element of $Zin(X)$ with $p(f) = -f$. Then we have $f \in ST(X)$.*

Proof. Write $a \equiv b$ if $a - b \in ST(X)$. If $p(f) = -f$, then f can be expressed as a linear combination of skew-right-commutative elements. In order to prove that the element $f \in ST(X)$, it is enough to show that

$$\overline{x_{i_1} \cdots x_{i_n}} \equiv 0. \tag{10}$$

We prove it by induction on n.

If $n = 2$, the proof is straightforward, and if $n = 3$, we have:

$$\overline{x_{i_1}x_{i_2}x_{i_3}} = \frac{1}{2}\left[\left[x_{i_1}, x_{i_2}\right], x_{i_3}\right] - \frac{1}{2}\left[\left[x_{i_1}, x_{i_3}\right], x_{i_2}\right].$$

Assuming that equation (10) holds for elements of degree less than n, we have

$$\overline{zx_{i_{k+1}} \cdots x_{i_n}} \equiv 0,$$

for any Lie element z whose degree is no more than k. Set $z := \overline{x_{i_1} \cdots x_{i_k}}$ and have

$$\overline{\overline{x_{i_1} \cdots x_{i_k}}x_{i_{k+1}} \cdots x_{i_n}} \equiv 0, \quad \text{for } 1 < k < n-1.$$

Hence,

$$\overline{x_{i_1} \cdots x_{i_k}x_{i_{k+1}} \cdots x_{i_n}} \equiv \overline{x_{i_1} \cdots x_{i_{k-2}}x_{i_k}x_{i_{k-1}}x_{i_{k+1}} \cdots x_{i_n}}.$$

Since the symmetric group S_{n-2} is generated by transpositions $(12), (23), \dots,$ $(n-3\ n-2)$, for any $\sigma \in S_{n-2}$ we have

$$\overline{x_{i_1} \cdots x_{i_n}} \equiv \overline{x_{\sigma(i_1)} \cdots x_{\sigma(i_{n-2})}x_{i_{n-1}}x_{i_n}}. \tag{11}$$

By (11) and Lemma 2.4 we have

$$\left[\overline{x_{i_1} \cdots x_{i_{n-1}}}, x_{i_n}\right] \equiv \overline{x_{i_1} \cdots x_{i_{n-2}}x_{i_{n-1}}x_{i_n}} - \overline{x_{i_1} \cdots x_{i_{n-2}}x_{i_{n-1}}x_{i_{n-2}}x_{i_n}} -$$

$$-(n-2)\overline{x_{i_1} \cdots x_{i_{n-3}}x_{i_n}x_{i_{n-2}}x_{i_{n-1}}} \equiv 0.$$

Also, can be obtained the following

$$\left[\overline{x_{i_1} \cdots x_{i_{n-2}}x_{i_n}}, x_{i_{n-1}}\right] \equiv$$

$$\overline{x_{i_1} \cdots x_{i_{n-2}}x_{i_n}x_{i_{n-1}}} - \overline{x_{i_1} \cdots x_{i_{n-3}}x_{i_n}x_{i_{n-2}}x_{i_{n-1}}} - (n-2)\overline{x_{i_1} \cdots x_{i_{n-3}}x_{i_n}x_{i_{n-2}}x_{i_{n-1}}} =$$

$$-\overline{x_{i_1}\cdots x_{i_{n-2}}x_{i_{n-1}}x_{i_n}}-\overline{x_{i_1}\cdots x_{i_{n-3}}x_{i_n}x_{i_{n-2}}x_{i_{n-1}}}-(n-2)\overline{x_{i_1}\cdots x_{i_{n-3}}x_{i_n}x_{i_{n-2}}x_{i_{n-1}}}.$$

Consider the sum of the above last two expressions and we have

$$[\overline{x_{i_1}\cdots x_{i_{n-1}}},x_{i_n}]+[\overline{x_{i_1}\cdots x_{i_{n-2}}x_{i_n}},x_{i_{n-1}}]\equiv$$

$$-(n-1)\overline{x_{i_1}\cdots x_{i_{n-3}}x_{i_{n-1}}x_{i_{n-2}}x_{i_n}}-(n-1)\overline{x_{i_1}\cdots x_{i_{n-3}}x_{i_n}x_{i_{n-2}}x_{i_{n-1}}}\equiv 0.$$

Thus

$$\overline{x_{i_1}\cdots x_{i_{n-3}}x_{i_{n-1}}x_{i_{n-2}}x_{i_n}}\equiv-\overline{x_{i_1}\cdots x_{i_{n-3}}x_{i_n}x_{i_{n-2}}x_{i_{n-1}}}.$$

In other words,

$$\overline{x_{i_1}\cdots x_{i_{n-3}}x_{i_{n-2}}x_{i_{n-1}}x_{i_n}}\equiv-\overline{x_{i_1}\cdots x_{i_{n-3}}x_{i_n}x_{i_{n-1}}x_{i_{n-2}}}. \qquad (12)$$

Set $u=x_{i_1}\cdots x_{i_{n-4}}$. By (11) and (12) we have

$$\overline{u\,x_{i_{n-3}}x_{i_{n-2}}x_{i_{n-1}}x_{i_n}}\equiv-\overline{u\,x_{i_{n-3}}x_{i_n}x_{i_{n-1}}x_{i_{n-2}}}\equiv\overline{u\,x_{i_n}x_{i_{n-3}}x_{i_{n-2}}x_{i_{n-1}}}\equiv$$

$$\overline{u\,x_{i_{n-1}}x_{i_n}x_{i_{n-3}}x_{i_{n-2}}}\equiv\overline{u\,x_{i_n}x_{i_{n-1}}x_{i_{n-3}}x_{i_{n-2}}}\equiv\overline{u\,x_{i_{n-2}}x_{i_n}x_{i_{n-1}}x_{i_{n-3}}}\equiv$$

$$\overline{u\,x_{i_{n-3}}x_{i_{n-2}}x_{i_n}x_{i_{n-1}}}\equiv-\overline{u\,x_{i_{n-3}}x_{i_{n-2}}x_{i_{n-1}}x_{i_n}}.$$

Hence,

$$\overline{u\,x_{i_{n-3}}x_{i_{n-2}}x_{i_{n-1}}x_{i_n}}\equiv 0$$

and this completes the proof.

Now we are ready to prove the main theorems of the section.

Theorem 2.2.8 *Let f be a Zinbiel element of $Zin(X)$. Then f is a Lie element if and only if $p(f)=-f$.*

Proof. It follows from Lemma 2.2.6 and Lemma 2.2.7.

Theorem 2.2.9 *The set of skew-rcom elements $\overline{x_\alpha}$, where $\alpha\in\Gamma$, forms base of $ST(X)$. Let $ST(X)_{m_1,\dots,m_q}$ be the homogenous part of $ST(X)$ generated by m_i generators x_i where $i=1,\dots,q$. Then*

$$dim\,ST(X)_{m_1,\dots,m_q}\;=\;\sum_{i<j}\frac{(n-2)!}{m_{1!}\cdots\;m_{q!}}m_im_j$$

where $n=m_1+\cdots+m_q$. In particlular, the multilinear part of $ST(X)$ has dimension $\frac{q!}{2}$.

Proof. Since a skew-rcom element defined as

$$\overline{x_{i_1} \cdots x_{i_n}} = x_{i_1} \cdots x_{i_{n-1}} x_{i_n} - x_{i_1} \cdots x_{i_{n-2}} x_{i_n} x_{i_{n-1}}$$

a difference of two base elements, any linear combination of skew-rcom elements is trivial, hence they are linearly independent in $Zin(X)$. By Lemma 2.2.6, any element of $ST(X)$ is a linear combination of skew-rcom elements. So we have proved that the set of skew-rcom elements, generated by set X, forms a base of $ST(X)$.

Let us count the number of skew-rcom elements of degree n generated by $x_1, \dots, x_q$ in which $x_1, \dots, x_q$ occur $m_1, \dots, m_q$ times, respectively. Consider skew-rcom elements whose last two elements are x_i, x_j for $i < j$. Then the number of such type of skew-rcom elements of degree n equals

$$\frac{(n-2)!}{m_{1!} \cdots (m_i - 1)! \cdots (m_j - 1)! \cdots m_q!} = \frac{(n-2)!}{m_1! \cdots m_q!} m_i m_j,$$

where $m_1 + \cdots + m_q = n$. Hence

$$\dim ST(X)_{m_1, \dots, m_q} = \sum_{i<j} \frac{(n-2)!}{m_1! \cdots m_q!} m_i m_j.$$

If $m_i = 1$ for all i, then $n = q$, and we obtain

$$\dim ST(X)_{1, \dots, 1} = \sum_{1 \le i < j \le q} (q-2)! = \frac{q!}{2}.$$

This completes the proof.

Corollary 2.2.10 *Let $a, b, c \in Zin(X)$. If b and c are Lie, then $abc - acb$ and $bc - cb$ are Lie.*

Proof. We present a proof of our Corollary for $abc - acb$. The case $bc - cb$ can be established in a similar way.

Let $a \in Zin(X)$ and suppose $b, c \in ST(X)$. Then by Theorem 2.2.9

$$b = \sum_{\alpha} \lambda_\alpha \overline{x_\alpha}, \qquad c = \sum_{\beta} \mu_\beta \overline{y_\beta}.$$

We have

$$abc - acb = a \sum_{\alpha} \lambda_\alpha \overline{x_\alpha} \sum_{\beta} \mu_\beta \overline{y_\beta} - a \sum_{\beta} \mu_\beta \overline{y_\beta} \sum_{\alpha} \lambda_\alpha \overline{x_\alpha} =$$

$$\sum_{\alpha,\beta} \lambda_\alpha \mu_\beta \left(a\, \overline{x_\alpha}\, \overline{y_\beta} - a\, \overline{y_\beta}\, \overline{x_\alpha}\,\right).$$

Let $x_\alpha = x_{i_1} \cdots x_{i_m}$ and $y_\beta = y_{j_1} \cdots y_{j_n}$. The cases, when at least one of m and n is equal to two, can be easily proved. Suppose $m, n \geq 3$. Set $u = x_{i_1} \cdots x_{i_{m-2}}$ and $v = y_{j_1} \cdots y_{j_{n-2}}$.

$$a\, \overline{x_\alpha}\, \overline{y_\beta} =$$

$$a\left(u\, x_{i_{m-1}}\, x_{i_m}\right)\left(v y_{j_{n-1}} y_{j_n}\right) - a\left(u\, x_{i_m}\, x_{i_{m-1}}\right)\left(v y_{j_{n-1}} y_{j_n}\right) -$$

$$a\left(u\, x_{i_{m-1}}\, x_{i_m}\right)\left(v y_{j_n} y_{j_{n-1}}\right) + a\left(u\, x_{i_m}\, x_{i_{m-1}}\right)\left(v y_{j_n} y_{j_{n-1}}\right) =$$

(by part **b** of Proposition 2.2.2)

$$\left(\left(a \sqcup\!\sqcup\, u\, x_{i_{m-1}}\right) x_{i_m} \sqcup\!\sqcup\, v y_{j_{n-1}}\right) y_{j_n} - \left(\left(a \sqcup\!\sqcup\, u x_{i_m}\right) x_{i_{m-1}} \sqcup\!\sqcup\, v y_{j_{n-1}}\right) y_{j_n} -$$

$$\left(\left(a \sqcup\!\sqcup\, u\, x_{i_{m-1}}\right) x_{i_m} \sqcup\!\sqcup\, v y_{j_n}\right) y_{j_{n-1}} + \left(\left(a \sqcup\!\sqcup\, u x_{i_m}\right) x_{i_{m-1}} \sqcup\!\sqcup\, v y_{j_n}\right) y_{j_{n-1}} =$$

(by part **c** of Proposition 2.2.2)

$$\left(\left(a \sqcup\!\sqcup\, u\, x_{i_{m-1}}\right) \sqcup\!\sqcup\, v y_{j_{n-1}}\right) x_{i_m} y_{j_n} + \left(\left(a \sqcup\!\sqcup\, u\, x_{i_{m-1}}\right) x_{i_m} \sqcup\!\sqcup\, v\right) y_{j_{n-1}} y_{j_n} -$$

$$\left(\left(a \sqcup\!\sqcup\, u\, x_{i_m}\right) \sqcup\!\sqcup\, v y_{j_{n-1}}\right) x_{i_{m-1}} y_{j_n} - \left(\left(a \sqcup\!\sqcup\, u\, x_{i_m}\right) x_{i_{m-1}} \sqcup\!\sqcup\, v\right) y_{j_{n-1}} y_{j_n} -$$

$$\left(\left(a \sqcup\!\sqcup\, u\, x_{i_{m-1}}\right) \sqcup\!\sqcup\, v y_{j_n}\right) x_{i_m} y_{j_{n-1}} - \left(\left(a \sqcup\!\sqcup\, u\, x_{i_{m-1}}\right) x_{i_m} \sqcup\!\sqcup\, v\right) y_{j_n} y_{j_{n-1}} +$$

$$\left(\left(a \sqcup\!\sqcup\, u\, x_{i_m}\right) \sqcup\!\sqcup\, v y_{j_n}\right) x_{i_{m-1}} y_{j_{n-1}} + \left(\left(a \sqcup\!\sqcup\, u\, x_{i_m}\right) x_{i_{m-1}} \sqcup\!\sqcup\, v\right) y_{j_n} y_{j_{n-1}} =$$

(by the definition of skew-right-commutative elements we obtain)

$$\overline{\left(\left(a \sqcup\!\sqcup\, u\, x_{i_{m-1}}\right) x_{i_m} \sqcup\!\sqcup\, v\right) y_{j_{n-1}} y_{j_n}} - \overline{\left(\left(a \sqcup\!\sqcup\, u\, x_{i_m}\right) x_{i_{m-1}} \sqcup\!\sqcup\, v\right) y_{j_{n-1}} y_{j_n}} +$$

$$\left(\left(a \sqcup\!\sqcup\, u\, x_{i_{m-1}}\right) \sqcup\!\sqcup\, v y_{j_{n-1}}\right) x_{i_m} y_{j_n} - \left(\left(a \sqcup\!\sqcup\, u\, x_{i_m}\right) \sqcup\!\sqcup\, v y_{j_{n-1}}\right) x_{i_{m-1}} y_{j_n} -$$

$$\left(\left(a \sqcup\!\sqcup\, u\, x_{i_{m-1}}\right) \sqcup\!\sqcup\, v y_{j_n}\right) x_{i_m} y_{j_{n-1}} + \left(\left(a \sqcup\!\sqcup\, u\, x_{i_m}\right) \sqcup\!\sqcup\, v y_{j_n}\right) x_{i_{m-1}} y_{j_{n-1}}.$$

By similar way one can have

$$a\,\overline{y_\beta}\,\overline{x_\alpha} =$$

$$\overline{\left((a \shuffle v\, y_{j_{n-1}})y_{j_n} \shuffle u\right) x_{i_{m-1}}\, x_{i_m}} - \overline{\left((a \shuffle v\, y_{j_n})y_{j_{n-1}} \shuffle u\right) x_{i_{m-1}}\, x_{i_m}} +$$

$$\left((a \shuffle v\, y_{j_{n-1}}) \shuffle u\, x_{i_{m-1}}\right) y_{j_n} x_{i_m} - \left((a \shuffle v\, y_{j_n}) \shuffle u\, x_{i_{m-1}}\right) y_{j_{n-1}}\, x_{i_m} -$$

$$\left((a \shuffle v\, y_{j_{n-1}}) \shuffle u\, x_{i_{m-1}}\right) y_{j_n} x_{i_{m-1}} + \left((a \shuffle v\, y_{j_n}) \shuffle u\, x_{i_m}\right) y_{j_{n-1}} x_{i_{m-1}}.$$

So

$$a\,\overline{x_\alpha}\,\overline{y_\beta} - a\,\overline{y_\beta}\,\overline{x_\alpha} =$$

(by part **a** of Proposition 2.2.2)

$$\overline{\left((a \shuffle u\, x_{i_{m-1}})x_{i_m} \shuffle v\right) y_{j_{n-1}} y_{j_n}} - \overline{\left((a \shuffle u\, x_{i_m})x_{i_{m-1}} \shuffle v\right) y_{j_{n-1}} y_{j_n}} +$$

$$\overline{\left((a \shuffle u\, x_{i_{m-1}}) \shuffle v y_{j_{n-1}}\right) x_{i_m} y_{j_n}} - \overline{\left((a \shuffle u\, x_{i_m}) \shuffle v y_{j_{n-1}}\right) x_{i_{m-1}} y_{j_n}} -$$

$$\overline{\left((a \shuffle u\, x_{i_{m-1}}) \shuffle v y_{j_n}\right) x_{i_m} y_{j_{n-1}}} + \overline{\left((a \shuffle u\, x_{i_m}) \shuffle v y_{j_n}\right) x_{i_{m-1}} y_{j_{n-1}}} -$$

$$\overline{\left((a \shuffle v\, y_{j_{n-1}})y_{j_n} \shuffle u\right) x_{i_{m-1}}\, x_{i_m}} + \overline{\left((a \shuffle v\, y_{j_n})y_{j_{n-1}} \shuffle u\right) x_{i_{m-1}}\, x_{i_m}}.$$

Hence by Theorem 2.2.8 we have $abc - acb \in ST(X)$ and this completes the proof.

2.3 Jordan elements in a free Zinbiel algebra

In this section, we demonstrate the proof of the Jordan criterion for elements in a free Zinbiel algebra $Zin(X)$. We provide an explicit formula for expanding Jordan bracketed elements in a free Zinbiel algebra.

Lemma 2.3.1

$$\left\{\left\{\cdots\left\{x_{i_1}, x_{i_2}\right\}\cdots\right\}, x_{i_n}\right\} = \sum_{\sigma \in S_n} x_{\sigma(i_1)} x_{\sigma(i_2)} \cdots\, x_{\sigma(i_n)}.$$

Proof. We prove it by induction on n. For the base of induction $n = 2$, we have $\{x_{i_1}, x_{i_2}\} = x_{i_1} x_{i_2} + x_{i_2} x_{i_1}$. Suppose that it is true for $n - 1$. Then

$$\left\{\left\{\cdots\left\{x_{i_1}, x_{i_2}\right\}\cdots\right\}, x_{i_n}\right\} =$$

$$\left\{\{\cdots\{x_{i_1},x_{i_2}\}\cdots\},x_{i_{n-1}}\right\}x_{i_n}+x_{i_n}\left\{\{\cdots\{x_{i_1},x_{i_2}\}\cdots\},x_{i_{n-1}}\right\}=$$

(by induction hypothesis)

$$\sum_{\sigma\in S_n}x_{\sigma(i_1)}x_{\sigma(i_2)}\cdots\, x_{\sigma(i_{n-1})}\,x_n+x_n\sum_{\sigma\in S_n}x_{\sigma(i_1)}x_{\sigma(i_2)}\cdots\, x_{\sigma(i_{n-1})}=$$

(by Proposition 2.2.1)

$$=\sum_{\sigma\in S_n}x_{\sigma(i_1)}x_{\sigma(i_2)}\cdots\, x_{\sigma(i_n)}.$$

This completes the proof.

Theorem 2.3.2 *Let f be a homogenous Zinbiel element of degree n in $Zin(X)$. Then f is a Jordan element if and only if $D(f)=n!\,f$. The algebra $J(X)$ is isomorphic to polynomial algebra $K[X]$.*

Proof. Recall that $A^{(+)}$ is associative and commutative algebra if A is Zinbiel. Any Jordan element in $Zin(X)$ can be written as linear combination of left-normed Jordan monomials in X by anti-commutators. Then the proof follows from Lemma 4.1 and definition of the map D.

Let $\varphi: K[X]\to J(X)$ be a canonical homomorphism from polynomial algebra generated by X to $J(X)$ defined as $x_{i_1}x_{i_2}\cdots x_{i_n}\mapsto\{\{\cdots\{x_{i_1},x_{i_2}\}\cdots\},x_{i_n}\}$. Then it is clear that $Ker\ \varphi$ is zero and therefore $K[X]$ and $J(X)$ are isomorphic. This completes the proof.

Denote by $J(X)_{m_1,\dots,m_q}$ the homogenous part of $J(X)$ generated by m_i generators x_i where $i=1,\dots,q$.

Corollary 2.3.3 *The dimension of the homogenous part $J(X)_{m_1,\dots,m_q}$ of $J(X)$ is equal to $dim\, J(X)_{m_1,\dots,m_q}=1$.*

Proof. It is an immediate consequence of Theorem 2.3.2.

2.4 Speciality of the free Tortkara algebra with two generators

In this section we prove that the free Tortkara algebra with two generators $T(\{x,y\})$ is special. As a corollary, we obtain the construction of a base of $T(\{x,y\})$ in terms of left-normed elements.

Lemma 2.4.1 *Let T_n be the n-th homogenous part of $T(\{x,y\})$. Then $T_{n+1}=T_nT_1$ for any n.*

Proof. Clearly, $T_{n+1}\supseteq T_nT_1$. We write $a\equiv b$ if $a-b\in T_nT_1$. We prove the statement by induction on degree n. We have

$$(ab)(cd)=\frac{1}{2}J(b,c,d)a-\frac{1}{2}J(a,c,d)b-\frac{1}{2}J(a,b,d)c+\frac{1}{2}J(a,b,c)d\equiv 0$$

This is the basis of induction for n. Suppose that our statement is true for fewer than $n > 4$. Let $C \in T_n$ and $C = A_k B_l$ where A_k and B_l are elements of $T(\{x, y\})$ whose degrees are k and l, respectively, and $k + l = n$. Now we consider induction on l. By induction on n we may assume that they are left-normed and write

$$C = A_k B_l = (A_{k-1} a_k)(B_{l-1} b_l)$$

where $a_k, b_l \in \{x, y\}$. Suppose $l = 2$ and $b_1 = x$, $b_2 = y$. Assume $a_k = x$. Then by Tortkara identity (4) and induction on n we have

$$C = (A_{k-1} x)(yx) = J(A_{k-1}, x, y)x \equiv 0.$$

Suppose that our statement is true for fewer than $l > 2$. We have

$$(A_{k-1} a_k)(B_{l-1} b_l) =$$

$$-(A_{k-1} a_k)(b_l \, B_{l-1}) =$$

(by identity (4))

$$(A_{k-1} B_{l-1})(b_l a_k) - J(A_{k-1}, a_k, b_l)B_{l-1} - J(A_{k-1}, B_{l-1}, b_l)a_k.$$

We note that by base of induction on l,

$$(A_{k-1} B_{l-1})(b_l a_k) \equiv 0 \text{ and } J(A_{k-1}, a_k, b_l)B_{l-1} \equiv 0.$$

By induction on n we have

$$J(A_{k-1}, B_{l-1}, b_l)a_k \equiv 0.$$

Hence, $(A_{k-1} a_k)(B_{l-1} b_l) \equiv 0$. This means that any element in $T(\{x, y\})$ can be written as a linear combination of left-normalized elements.

Let $T(X)$ be a free Tortkara algebra generated by a set X.

Theorem 2.4.2 *The free Tortkara algebra $T(\{x, y\})$ is special.*

Proof. It is sufficient to show that algebras $T(\{x, y\})$ and $ST(\{x, y\})$ are isomorphic. Let φ be a natural homomorphism from $T(\{x, y\})$ to $ST(\{x, y\})$. By Lemma 2.4.1 the vector space $T(\{x, y\})$ is spanned by the set of left-normed elements. We note that number of left-normed elements in two generators is equal to the number of skew-rcom elements in two generators. Suppose that the kernel of φ is not zero. Then we have a linear combination of skew-rcom elements which is zero in $ST(\{x, y\})$. It contradicts to the first part of Theorem 2.2.9. Therefore, $Ker\varphi = (0)$. This completes the proof.

Corollary 2.4.3. *Set of left-normed elements forms a base of $T(\{x, y\})$.*

Proof. It is an immediate consequence of Theorem 2.4.2.

2.5 Speciality of homomorphic images of $ST(\{x, y\})$

The next theorem is an analogue of Cohn's theorem on the speciality of homomorphic images of special Jordan algebras in two generators Lemma 1.3.4 [7, p. 255]) *Let α be an ideal of free special Jordan algebra $SJ(X)$ and $\{\alpha\}$ is an ideal of free special associative algebras generated by the set α. Then $SJ(X)/\alpha$ is a special Jordan algebra if and only if $\{\alpha\} \cap SJ(X) \subseteq \alpha$.*

Theorem 2.5.1 *Any homomorphic image of a free special Tortkara algebra with two generators is special. For the three generators case, this statement is not true: a homomorphic image of special Tortkara algebra with three generators might be non-special.*

Let α be an ideal of $ST(X)$. By Cohn's criterion (Theorem 2.2 of [7, p. 255]) $ST(X)/\alpha$ is special if and only if $\alpha \cap ST(X) \subseteq \alpha$ where α is the ideal of $Zin(X)$ generated by the set α.

Proof of Theorem 2.5.1. Assume that g_i $(i \in I)$ are generators of the ideal α. It is clear that if $\overline{xy} \in \alpha$ then $ST(\{x, y\})/\alpha$ is special.

Therefore, by Theorem 2.2.8 we can assume that each element g_i has a form $\overline{f_i xy}$ for some $f_i \in Zin(\{x, y\})$.

Let w be a non-zero element of $\alpha \cap ST(\{x, y\})$. Then $p(w) = -w$ and w is a linear combination of left-normed monomials in $x, y, g_i (i \in I)$ such that each monomial is linear by at least one generator of α. Let $b_1 \cdots b_n$ be a term of w in the linear combination. To prove the statement we consider two cases, depending on what position a generator appear in $b_1 \cdots b_n$.

Case 1. Suppose that generators of α appear only in the first $n-2$ positions in $b_1 \cdots b_n$. Then write all b_i's in terms of elements of X. Since $w \in ST(\{x, y\})$, w must have the term $p(b_1 \cdots b_n)$ with opposite sign. Hence $\overline{b_1 \cdots b_n} \in \alpha$.

Case 2. Suppose that generators of α appear in either $n-1$-th or n-th positions in $b_1 \cdots b_n$, (a generator of α may appear in the first $n-2$-positions), namely,

$$b_{n-1} = x \text{ and } b_n = \overline{f_i xy}, \text{ or } b_{n-1} = \overline{f_i xy} \text{ and } b_n = x$$

for some i. If generators of α appear in both $n-1$-th and n-th positions of $b_1 \cdots b_n$, then write one of them in terms of $x's$ and $y's$. We also express $b_1, \dots, b_{n-2}$ in terms of $x's$ and $y's$, therefore we can assume that $b_1, \dots, b_{n-2} \in X$. Let us denote $b_1 \cdots b_{n-2}$ by u.

Now we show that if $ux\overline{f_i xy}$ is a term of w then w has the term $u\overline{f_i xy}x$ with opposite sign.

We have

$$ux\overline{f_i xy} = R_1 + R_2 - R_3,$$

where

$$R_1 = \overline{f_i uxxy} + \overline{uf_i xxy} + \overline{uxf_i xy},$$

$$R_2 = f_i uxxy + f_i xuxy + uf_i xxy,$$

$$R_3 = f_i uyxx + f_i yuxx + uf_i yxx.$$

By Theorem 2.2.8, R_1 is Lie, but R_2, R_3 are not Lie. Since $w \in ST(\{x,y\})$, the term R_3 should be cancelled and for each term $s \in f_i uxxy, f_i xuxy, uf_i xxy$ of R_2, w must have terms s or $p(s)$ with opposite sign to cancel s or have $\overline{s}$. Therefore, w must have some terms in which g_i $(i \in I)$ appear in either $n-1$-th or n-th positions. These kind of terms are generated by u, f, x, x, y. Then all possibilities of such types are $u\overline{f_i xy}x, f_i x\overline{uxy}$ and $f_i\overline{uxy}x$. We have

$$u\overline{f_i xy}x = f_i uxyx + f_i xuyx + uf_i xyx - f_i uyxx - f_i yuxx - uf_i\, yxx,$$

$$f_i x\overline{uxy} = \overline{f_i uxxy} + \overline{f_i xuxy} + \overline{uf_i xxy} + f_i uxxy + uf_i xxy + uxf_i xy -$$

$$-f_i uyxx - uf_i yxx - uyf_i xx,$$

$$f_i\overline{uxy}x = f_i uxyx + uf_i xyx + uxf_i yx - f_i uyxx - uf_i yxx - uyf_i xx.$$

We see that the element $f_i yuxx$ is a term of only $u\overline{f_i x}x$ and moreover,

$$ux\overline{f_i xy} - u\overline{f_i xy}x = \overline{f_i uxxy} + \overline{uf_i xxy} + \overline{uxf_i xy} + \overline{f_i uxxy} +$$

$$\overline{f_i xuxy} + \overline{uf_i xxy} \in ST(\{x,y\}).$$

Therefore, w has the term $u\overline{f_i x}yx$ with opposite sign. Since $\overline{f_i xy}$ is a generator of α, and by Corollary 2.2.3

$$ux\overline{f_i xy} - u\overline{f_i xy}x \in \alpha.$$

Hence $b_1 \cdots b_n - b_1 \cdots b_{n-2} b_n b_{n-1} \in \alpha$. If $x\overline{f_i xy}$ is a nonzero term of w, then by similar way one can show that w must have term $\overline{f_i x}yx$.

So we obtain $w \in \alpha$. It follows $\alpha \cap ST(\{x,y\}) \subseteq \alpha$. Hence by Cohn's criterion $ST(\{x,y\})/\alpha$ is special.

Now we show that a homomorphic image of $ST(\{x,y,z\})$ may be not special. Let α be an ideal of $ST(\{x,y,z\})$ generated by elements

$$g_1 = \overline{yyz}, \qquad g_2 = \overline{yxz}, \qquad g_3 = \overline{yxy}.$$

Consider an element

$$w = \overline{xyyz} - \overline{yyxz} + \overline{zyxy}.$$

Then

$$w = xg_1 - yg_2 + zg_3.$$

It follows

$$w \in ST(\{x, y, z\}).$$

One can easily check that there are no $\lambda_1, \lambda_2, \lambda_3 \in K$ so that

$$w = \lambda_1[x, g_1] + \lambda_2[y, g_2] + \lambda_3[z, g_3].$$

Then $w \notin \alpha$. Hence by Cohn's criterion, $ST(\{x, y, z\})/\alpha$ is not special.

Corollary 2.5.2 *Any Tortkara algebra with two generators is special.*

Proof. It follows from Theorem 2.4.2 and Theorem 2.5.1.

This result is an analogue of Shirshov theorem for Jordan algebras [51, p. 84].

2.6 Some remarks and open questions

1. Let $A = C[x]$ be an algebra with multiplication

$$a \star b = b \int_0^x \left(\int_0^x a\, dx\right) dx. \tag{13}$$

$(A,\star)$ is not a Zinbiel algebra. This algebra $C[x]$ with multiplication (13) was considered in [1]. It was proved that it satisfies the following identities

$$\begin{gathered} a \star (b \star c) - b \star (a \star c) = 0, \\ ([a, b], c, d) + ([b, c], a, d) + ([c, a], b, d) = 0 \end{gathered} \tag{14}$$

where $(a, b, c) = a \star (b \star c) - (a \star b) \star c$. Moreover, it was proved that algebra A with respect to commutator $[a, b]_\star = a \star b - b \star a$ is a Tortkara algebra. A question on speciality of $(A, [,]_\star)$ was posed.

We show that answer is positive. Let $B = C[x]$ be an algebra with multiplication

$$a \diamond b = b \int_0^x \left(\int_0^x a dx\right) dx + \left(\int_0^x a dx\right)\left(\int_0^x b dx\right).$$

Then $(B,\diamond)$ is a Zinbiel algebra. For $\diamond$ multiplication we define commutator $[a, b]_\diamond = a \diamond b - b \diamond a$. Note that $[a, b]_\star = [a, b]_\diamond$. So $A^{(-)}$ is isomorphic to $B^{(-)}$. Hence $(A, [,]_\star)$ is special.

2. It is shown in [1] that an algebra with identities (14) is not Zinbiel but under the commutator product is Tortkara. What about speciality of these algebras?

3. Let k_m be kernel of the natural homomorphism from free Tortkara algebra to free special Tortkara algebra on m generators. An element of the ideal k_m is called a s-identity. We showed that $k_2 = (0)$. Are there s-identities for $m > 2$?

4. Is it true the analogue of Lemma 2.4.1 for $m > 2$ generators? Whenever it is valid for m generators, it immediately follows speciality of $T(\{x_1, \dots, x_m\})$, in particular, $k_m = (0)$.

3 NILPOTENT ASSOSYMMETRIC ALGEBRAS

In this chapter, we mainly study assosymmetric algebras of finite class and study commutator ideals of assosymmetric algebras. We show that some of the properties for associative algebras also hold for assosymmetric algebras, namely, for such properties associativity is not necessary and can be replaced by left-symmetry and right-symmetry. The results of this chapter were published in [41 – 43, 49].

3.1 Commutator ideals of assosymmetric algebras

We begin with some basic facts on Lie-admissible algebras. The results of this section were published in [49].

Let A be an arbitrary Lie-admissible algebra over a given field $\mathbb{F}$. We define

$$[a, b] = ab - ba$$

for all a and $b \in A$. For all subspaces B, C, D of A, we define

$$[B, C] = \mathrm{span}\{[b, c] | b \in B, c \in C\}, \quad BC = \mathrm{span}\{bc | b \in B, c \in C\}$$

and

$$(B, C, D) = \mathrm{span}\{(b, c, d) | \in B, c \in C, d \in D\},$$

where the associator (a, b, c) means $(ab)c - a(bc)$. We call a space $V \subseteq A$ a *Lie ideal* of A if we have $[V, A] \subseteq V$. Finally, for all subspaces A and B of A, we define

$$A \circ B = \mathrm{Id}([A, B]),$$

that is, the ideal of A generated by $[A, B]$. Following the idea of Jennings [19, p. 341], we call $A \circ B$ the *commutator ideal* of A and B. We clearly have $A \circ B = B \circ A$.

Equipped with the notion of commutator ideals, we are now able to recall the notion of central chains of ideals of a Lie-admissible algebra A.

Let

$$A = A_1 \supseteq A_2 \supseteq \cdots \supseteq A_m \supseteq A_{m+1} = (0) \tag{15}$$

be a chain of ideals of A. Such a chain is called a *central chain of ideals* if we have

$$A \circ A_i \subseteq A_{i+1} \quad (i = 1,2,\dots, m). \tag{16}$$

We shall soon see that Novikov algebras, bicommutative algebras and assosymmetric algebras which possess central chains of ideals have special properties; we investigate some of them by considering a particular central chain:

Definition 3.1.1 *For every Lie-admissible algebra A we form a series of ideals*

$$H_1 := A,\ H_{i+1} := H_i \circ A \quad for \quad i \geq 1. \tag{17}$$

We say that A is of *finite class* if $H_n = (0)$ for some positive integer n. For the minimal integer n such that $H_n = (0)$, we call $n-1$ the *class* of A, and call

$$A = H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = (0) \tag{18}$$

the *lower central chain* of A. To avoid too many repetitions, we shall fix the notation of H_i for all $i \geq 1$.

With the notations of (16) and (18), it is straightforward to show that $H_i \subseteq A_i$ by induction on i.

The next subsection provides us with a description of commutator ideals of assosymmetric algebras.

3.1.1 Assosymmetric algebras of finite class

The aim of this subsection is to study assosymmetric algebras of finite class. Recall that for all x, y, z in an algebra A, the associator (x, y, z) means $(xy)z - x(yz)$. So in every assosymmetric algebra A, we have $(y, x, z) = (x, y, z) = (x, z, y)$ for all $x, y, z \in A$.

It is proved in [22, p. 984] that for all x, y, z, w in an assosymmetric algebra A, we have

$$([x, y], z, w) = 0 \quad if \quad \mathrm{char}(\mathbb{F}) \neq 2{,}3. \tag{19}$$

By the same technique developed in [22, p. 983], we obtain some more identities as follows when $\mathrm{char}(\mathbb{F}) = 2$ or 3.

Lemma 3.1.2 *For all x, y, z, w in an assosymmetric algebra A, we have*

$$(x, y, z) = -[x, y]z + x[y, z] + [xz, y]; \tag{20}$$

$$([w, x], y, z) = [w, (x, y, z)] + [x, (w, y, z)] \quad if \quad \mathrm{char}(\mathbb{F}) = 2; \tag{21}$$

$$[xy, z] = -[yz, x] - [zx, y] \quad if \quad \mathrm{char}(\mathbb{F}) = 3; \tag{22}$$

Proof. (i) Proof of identity (20). Note that

$$(x, y, z) = -(x, y, z) + (y, x, z) + (x, z, y)$$

$$= -(xy)z + x(yz) + (yx)z - y(xz) + (xz)y - x(zy)$$

$$= -(xy)z + (yx)z + x(yz) - x(zy) + (xz)y - y(xz)$$

$$= -[x, y]z + x[y, z] + [xz, y].$$

The proof of identity (20) is completed.

(ii) Proof of identity (21). Following [22, p. 984], we define

$$f(w, x, y, z) = (wx, y, z) - x(w, y, z) - (x, y, z)w.$$

Then it is obvious that

$$f(w, x, y, z) = f(w, x, z, y). \tag{23}$$

We also note that [22, p. 984] in any algebra we have

$$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z. \tag{24}$$

By identity (24), we deduce

$$f(w, x, y, z) + f(z, w, x, y) =$$

$$(wx, y, z) - x(w, y, z) - (x, y, z)w + (zw, x, y) - w(z, x, y) - (w, x, y)z =$$

$$(wx, y, z) - x(w, y, z) - (x, y, z)w + (zw, x, y) -$$

$$(wx, y, z) + (w, xy, z) - (w, x, yz =$$

$$-x(w, y, z) - (x, y, z)w + (zw, x, y) + (w, xy, z) - (w, x, yz) =$$

$$-x(w, y, z) - (x, y, z)w + (xy, z, w) - (x, yz, w) + (x, y, zw) = 0.$$

Combining this with identity (23), we obtain

$$f(w, x, y, z) = -f(z, w, x, y) = f(y, z, w, x), \tag{25}$$

and thus

$$f(w, x, y, z) = f(y, z, w, x) = f(y, z, x, w) = f(x, w, y, z). \tag{26}$$

Therefore, if $\mathrm{char}(\mathbb{F}) = 2$, we obtain

$$0 = 2f(w, x, y, z)$$

$$= f(w, x, y, z) + f(x, w, y, z)$$

$$= (wx, y, z) + x(w, y, z) + (x, y, z)w +$$

$$(xw, y, z) + w(x, y, z) + (w, y, z)x$$

$$= ([w, x], y, z) + [w, (x, y, z)] + [x, (w, y, z)].$$

The proof of identity (21) is completed.

(iii) Proof of identity (22). If $\mathrm{char}(\mathbb{F}) = 3$, then we have

$$[xy, z] + [yz, x] + [zx, y] = (xy)z - z(xy) + (yz)x - x(yz) + (zx)y - y(zx)$$

$$= (x, y, z) + (y, z, x) + (z, x, y) = 3(x, y, z) = 0.$$

Identity (22) follows immediately.

Now we begin to study associators involving Lie ideals of an assosymmetric algebra A.

Lemma 3.1.3 *Let* B *and* C *be Lie ideals of* A. *Then the following statements are true:*

(i) *For all* $x \in C$, $y, z \in B$, $(y, x, z) \in A[C, B] + [C, B]$; *In particular,* (y, x, z) *is contained in the ideal of A generated by* $[C, B]$;

(ii) $A \circ B = [B, C] + A[B, C] = [B, C] + [B, C]A$.

Proof. (i) By identity (20), we deduce

$$(y, x, z) = -[y, x]z + y[x, z] + [yz, x] =$$

$$-[[y, x], z] - z[y, x] + y[x, z] + [yz, x] \in A[C, B] + [C, B].$$

The proof is completed.

(ii) Clearly, $[B, C]$ is a Lie ideal of A. It follows that

$$[B, C]A \subseteq A[B, C] + [[B, C], A] \subseteq A[B, C] + [B, C].$$

In particular, $[B, C] + A[B, C]$ is an ideal of A if and only if so does $[B, C] + [B, C]A$.

By (i), for all $x, y \in A$, $a \in B$, $b \in C$, we have

$$(x, [a, b], y) \in [[B, C], A] + A[[B, C], A] \subseteq A[B, C] + [B, C].$$

It follows that

$$x(y[a, b]) = (xy)[a, b] - (x, y, [a, b]) =$$

$$(xy)[a, b] - (x, [a, b], y) \in A[A, B] + [A, B].$$

Therefore, we deduce

$$(x[a,b])y = (x,[a,b],y) + x([a,b]y) =$$

$$(x,[a,b],y) + x(y[a,b]) + x[[a,b],y] \in A[B,C] + [B,C].$$

The proof is completed.

Corollary 3.1.4 *Let $\{A_p \mid p \geq 1\}$ be a family of ideals of A with $A \circ A_p \subseteq A_{p+1}$ for every $p \geq 1$. Then for all $a \in A_p$, for all $x, y \in A$, we have $(x,a,y) \in A_{p+1}$.*

Proof. By Lemma 3.1.3, we have

$$(x,a,y) \in [A_p, A] + A[A_p, A] = A \circ A_p \subseteq A_{p+1}.$$

The proof is completed.

Let

$$A = A_1 \supseteq A_2 \supseteq \cdots \supseteq A_m \supseteq A_{m+1} = (0) \tag{27}$$

be a central chain of ideals of A. And let H_i $(i \geq 1)$ be as in Definition 3.1.1. When A is assosymmetric, we have the following analogues as those for associative algebras. Again, as the associativity does not hold, new techniques are necessary.

Lemma 3.1.5 *Let A be an assosymmetric algebra. Then $H_p A_q \subseteq A_{p+q-1}$, $A_q H_p \subseteq A_{p+q-1}$, $[H_p, A_q] \subseteq A_{p+q}$ and $(H_p, A_q, A) \subseteq A_{p+q}$. In particular, we have $H_p H_q \subseteq H_{p+q-1}$.*

Proof. Since $H_p \subseteq A_p$, $A_{p+q} \subseteq A_{p+q-1}$ and $A_q H_p \subseteq [A_q, H_p] + H_p A_q$, it suffices to prove $H_p A_q \subseteq A_{p+q-1}$, $[H_p, A_q] \subseteq A_{p+q}$ and $(H_p, A_q, A) \subseteq A_{p+q}$.

We use induction on p to prove these claims. For $p = 1$, we have $H_1 A_q \subseteq A_q$, $[H_1, A_q] \subseteq A_{q+1}$ and by Corollary 3.1.4, we obtain

$$(H_1, A_q, A) \subseteq (A, A_q, A) \subseteq A \circ A_q \subseteq A_{q+1}.$$

Now we assume $p \geq 2$. If $h_p = [h_{p-1}, x]$ for some $h_{p-1} \in H_{p-1}$ and $x \in A$, then for every $a \in A_q$, by induction hypothesis, we have

$$h_p a = [h_{p-1}, x]a \overset{(20)}{=} -(h_{p-1}, x, a) + h_{p-1}[x,a] + [h_{p-1}a, x]$$

$$\in (H_{p-1}, A_q, A) + H_{p-1}A_{q+1} + [H_{p-1}A_q, A] \subseteq A_{p+q-1}.$$

By the Jacobi identity and induction hypothesis, we obtain

$$[h_p, a] = [[h_{p-1}, x], a] = [[h_{p-1}, a], x] + [h_{p-1}, [x, a]]$$

$$\in [A_{p+q-1}, A] + [H_{p-1}, A_{q+1}] \subseteq A_{p+q}.$$

We continue to show $([h_{p-1},x],a,w) \in A_{p+q}$ for every $w \in A$. There are several cases to discuss depending on the characteristic of the field. If $\mathrm{char}(\mathbb{F}) \neq 2,3$, then by identity (19), we have $([h_{p-1},x],a,w) = 0 \in A_{p+q}$. If $\mathrm{char}(\mathbb{F}) = 2$, then by identity (21) and by Corollary 3.1.4, we have

$$([h_{p-1},x],a,w) = [h_{p-1},(x,a,w)] + [x,(h_{p-1},a,w)]$$

$$\in [H_{p-1},A_{q+1}] + [A,A_{p+q-1}] \subseteq A_{p+q}.$$

If $\mathrm{char}(\mathbb{F}) = 3$, then by identities (20) and (22) and by the above reasoning, we have

$$([h_{p-1},x],a,w) =$$

$$= -[[h_{p-1},x],a]w + [h_{p-1},x][a,w] + [[h_{p-1},x]w,a]$$

$$= -[[h_{p-1},x],a]w + [h_{p-1},x][a,w] - [wa,[h_{p-1},x]] - [a[h_{p-1},x],w]$$

$$\in A_{p+q} + [h_{p-1},x]A_{q+1} + [A_{p+q-1},A] \subseteq A_{p+q}.$$

Now we prove for the case when $p \geq 2$ and $h_p = [h_{p-1},x]y$ for some elements $h_{p-1} \in H_{p-1}$ and $x,y \in A$. By the above reasoning and by the right-symmetric identity, we have

$$h_p a = ([h_{p-1},x],y,a) + [h_{p-1},x](ya) = ([h_{p-1},x],a,y) + [h_{p-1},x](y \in A_{p+q-1}.$$

By identity (20) and by the above reasoning, we obtain

$$[h_p,a] = [[h_{p-1},x]y,a] = ([h_{p-1},x],a,y) + [[h_{p-1},x],a]y - [h_{p-1},x][a,y]$$

$$\in A_{p+q} + [h_{p-1},x]A_{q+1} \subseteq A_{p+q}.$$

Finally, by the above reasoning and by the induction hypothesis, for every $w \in A$, we deduce

$$([h_{p-1},x]y,a,w) \overset{(20)}{=} -[[h_{p-1},x]y,a]w + ([h_{p-1},x]y)[a,w] + [([h_{p-1},x]y)w,a]$$

$$\in [H_p,A_q]A + H_pA_{q+1} + [H_p,A_q] \subseteq A_{p+q}.$$

The proof is completed.

Theorem 3.1.6 *Let A be an assosymmetric algebra of finite class. Then $A \circ A$ is nilpotent of nilpotent index less or equal to the class of A.*

Proof. By Lemma 3.1.5 and by similar reasoning as the proof for Theorem 2.5 in [49], we obtain the description for assosymmetric algebras that generalizes the corresponding result of associative algebras.

3.1.2 Products of commutator ideals of assosymmetric algebras

The aim of this subsection is to study products of commutator ideals of an arbitrary assosymmetric algebra A over a field F such that $\mathrm{char}(\mathbb{F}) \neq 2,3$.

Let A be an assosymmetric algebra. We define

$$A_{[1]} = A \quad \text{and} \quad A_{[i+1]} = [A, A_{[i]}] \text{ for all } i \geq 1.$$

We call $\mathrm{Id}(A_{[i]})$ the *ith commutator ideal* of A. And the algebra A is called *Lie nilpotent* if $A_{[i]} = (0)$ for some integer i. We shall prove that $\mathrm{Id}(A_{[i]})\mathrm{Id}(A_{[j]}) \subseteq \mathrm{Id}(A_{[i+j-1]})$ if i is odd or j is odd, which generalizes the corresponding result [48, p. 300] for associative algebras.

For all $x, y \in A$, we define

$$x * y = xy + yx \quad \textit{and} \quad [x, y] = xy - yx.$$

The main difference in the above-mentioned result between associative algebras and assosymmetric algebras is the proof of the following lemma.

Lemma 3.1.7 *Let A be an assosymmetric algebra over a field $\mathbb{F}$ such that $char(\mathbb{F}) \neq 2,3$. For every positive odd integer j, we have $[Id(A_{[j]}), A] \subseteq A_{[j+1]}$. Moreover, we have $[Id(A_{[j]}), A_{[i]}] \subseteq A_{[i+j]}$.*

Proof. For all $x, y, z \in A$, we have $[x, [y, z]] = [[x, y], z] - [[x, z], y]$. So the second claim follows immediately from the first one. We use induction on j to prove the lemma. For $j = 1$, the claim follows immediately by the definition of $A_{[2]}$ and by the above reasoning if $i \geq 2$. Now we assume that j is an odd integer such that $j \geq 3$. For all $x, y, z, u, v \in A$, it suffices to show $[x[y, [z, u]], v] \in A_{[j+1]}$ if $u \in A_{[j-2]}$. By assumption, we have $\mathrm{char}(\mathbb{F}) \neq 2$, so we have

$$x[y, [z, u]] = \tfrac{1}{2}([x, [y, [z, u]]] + x * [y, [z, u]]).$$

So in order to show $[x[y, [z, u]], v] \in A_{[j+1]}$, it suffices to prove $[x * [y, [z, u]], v] \in A_{[j+1]}$. The idea of the proof is to show that $[x * [y, [z, u]], v]$ is sort of skew-symmetric. More precisely, we shall prove that, if one of x, y, z, u, v lies in $A_{[j-2]}$ then

$$[x * [y, [z, u]], v] \equiv [x * [z, [u, y]], v] \equiv [x * [u, [y, z]], v] \; mod \; A_{[j+1]}.$$

Since A is an assosymmetric algebra, by identity (19), we have

$$(x, y, [z, u]) = (x, [z, u], y) = ([z, u], x, y) = 0 =$$

$$(xy)[z,u] - x(y[z,u]) = (x[z,u])y - x([z,u]y),$$

and thus

$$\begin{aligned}
& x * [y,[z,u]] + y * [x,[z,u]] \\
= \quad & x(y[z,u] - [z,u]y) + (y[z,u] - [z,u]y)x \\
& +y(x[z,u] - [z,u]x) + (x[z,u] - [z,u]x)y \\
= \quad & (xy)[z,u] - x([z,u]y) + y([z,u]x) - [z,u](yx) \\
& +(yx)[z,u] - y([z,u]x) + x([z,u]y) - [z,u](xy) \\
= \quad & (xy + yx)[z,u] - [z,u](xy + yx) \\
= \quad & [x * y,[z,u]].
\end{aligned}$$

Let assume that one of x, y, z, u, v lies in $A_{[j-2]}$. Since $[A_{[i]}, A_{[t]}] \subseteq A_{[i+t]}$, by the induction hypothesis, we obtain that $[[x * y,[z,u]], v]$ lies in $A_{[j+1]}$, and thus we deduce

$$[x * [y,[z,u]], v] \equiv -[y * [x,[z,u]], v] \; mod \; A_{[j+1]}. \tag{28}$$

Similarly, we have

$$\begin{aligned}
& [x * [y,[z,u]], v] + [v * [y,[z,u]], x] \\
& = (x[y,[z,u]] + [y,[z,u]]x)v - v(x[y,[z,u]] + [y,[z,u]]x) \\
& +(v[y,[z,u]] + [y,[z,u]]v)x - x(v[y,[z,u]] + [y,[z,u]]v) \\
& = x([y,[z,u]]v) + [y,[z,u]](xv) - (vx)[y,[z,u]] - v([y,[z,u]]x) \\
& +v([y,[z,u]]x) + [y,[z,u]](vx) - (xv)[y,[z,u]] - x([y,[z,u]]v) \\
& = [y,[z,u]](x * v) - (x * v)[y,[z,u]] \\
& = -[x * v,[y,[z,u]]].
\end{aligned}$$

Again, since one of x, y, z, u, v lies in $A_{[j-2]}$, by the induction hypothesis, we easily obtain that $[x * v,[y,[z,u]]]$ lies in $A_{[j+1]}$, and thus we deduce

$$[x * [y, [z, u]], v] \equiv -[v * [y, [z, u]], x] \ mod\ A_{[j+1]}. \tag{29}$$

On the other hand, by identity (28), we have

$$[x * [y, [z, u]], v] = -[x * [z, [u, y]], v] - [x * [u, [y, z]], v]$$

$$\equiv [z * [x, [u, y]], v] + [u * [x, [y, z]], v] \ modA_{[j+1]};$$

Interchanging x and y in the above equation, we obtain

$$[y * [x, [z, u]], v] \equiv [z * [y, [u, x]], v] + [u * [y, [x, z]], v] \ mod\ A_{[j+1]}.$$

So by the above two Equations and by the Jacobi identity, we deduce

$$2[x * [y, [z, u]], v] =$$

$$= [x * [y, [z, u]], v] - [y * [x, [z, u]], v]$$

$$\equiv [z * [x, [u, y]], v] + [u * [x, [y, z]], v] - [z * [y, [u, x]], v] - [u * [y, [x, z]], v]$$

$$\equiv [z * [u, [x, y]], v] - [u * [z, [x, y]], v]$$

$$\equiv 2[z * [u, [x, y]], v] \ mod\ A_{[j+1]}.$$

Since $\mathrm{char}(\mathbb{F}) \neq 2$, we obtain

$$[x * [y, [z, u]], v] \equiv [z * [u, [x, y]], v] \ mod\ A_{[j+1]}. \tag{30}$$

Therefore, in the vector space $A/A_{[j+1]}$, we have

$$\begin{aligned} &\left[x * \left[y, [z, u]\right], v\right] \overset{(28)}{\equiv} -\left[y * \left[x, [z, u]\right], v\right] \overset{(28)(29)}{\equiv} \\ &[v * [x, [z, u]], y] \overset{(28)(29)(28)}{\equiv} -[x * [v, [z, u]], y], \end{aligned} \tag{31}$$

$$\begin{aligned} &\left[x * \left[y, [z, u]\right], v\right] \overset{(30)}{\equiv} \left[z * \left[u, [x, y]\right], v\right] \overset{(30)(29)}{\equiv} \\ &-[v * [u, [x, y]], z] \overset{(30)(29)(30)}{\equiv} [x * [y, [v, u]], z], \end{aligned} \tag{32}$$

and thus

$$[x*[y,[z,u]],v] \equiv -[x*[y,[u,z]],v] \overset{(32)}{\equiv} [x*[y,[v,z]],u] \equiv -[x*[y,[z,v]],u]. \tag{33}$$

Therefore, we deduce

$$[x*[y,[z,u]],v] \overset{(31)}{\equiv} -[x*[v,[z,u]],y] \overset{(31)(33)}{\equiv} [x*[v,[z,y]],u]$$

$$\overset{(31)(33)(31)}{\equiv} -[x*[u,[z,y]],v] \equiv [x*[u,[y,z]],v] \; mod \; A_{[j+1]}.$$

It follows that

$$[x*[y,[z,u]],v] \equiv [x*[u,[y,z]],v] \equiv [x*[z,[u,y]],v] \; mod \; A_{[j+1]}.$$

Finally, since A is Lie-admissible, we obtain

$$3[x*[y,[z,u]],v] \equiv$$

$$[x*[y,[z,u]],v] + [x*[u,[y,z]],v] + [x*[z,[u,y]],v] \equiv 0 \; mod \; A_{[j+1]}.$$

Since char($\mathbb{F}$) $\neq 3$, we have $[x*[y,[z,u]],v] \in A_{[j+1]}$. The proof is completed.

We conclude the section with the main result of this subsection, which generalizes the corresponding property of associative algebras. The results also published in [49].

Theorem 3.1.8 *Let A be an assosymmetric algebra. Then we have $Id(A_{[i]})Id(A_{[j]}) \subseteq Id(A_{[i+j-1]})$ if i or j is odd.*

Proof. If $i = 1$ or $j = 1$, then clearly we have $Id(A_{[i]})Id(A_{[j]}) \subseteq Id(A_{[i+j-1]})$. Now we assume $i \geq 2$ and $j \geq 2$. Then by Lemma 3.1.3 (ii) and by identity (19), we have

$$Id(A_{[i]})Id(A_{[j]}) = (A_{[i]} + AA_{[i]})(A_{[j]} + A_{[j]}A)$$

$$\subseteq A_{[i]}A_{[j]} + A(A_{[i]}A_{[j]}) + (A_{[i]}A_{[j]})A + A(A_{[i]}A_{[j]})A.$$

So it suffices to show $A_{[i]}A_{[j]} \subseteq \mathrm{Id}(A_{[i+j-1]})$ if one of i and j is odd. Since

$$A_{[i]}A_{[j]} \subseteq [A_{[i]}, A_{[j]}] + A_{[j]}A_{[i]} \subseteq A_{[i+j]} + A_{[j]}A_{[i]},$$

we may assume that j is odd and thus we may assume $j \geq 3$ and $i \geq 2$. For all $x \in A$, $y \in A_{[i-1]}$ and $z \in A_{[j]}$, by identity (19) and by Lemma 3.1.7, we have

$$[x,y]z = (xy)z - (yx)z = x(yz) - y(xz) = x(yz) - x(zy) + x(zy) - y(xz)$$

$$= x(yz) - x(zy) + (xz)y - y(xz) = x[y,z] + [xz,y]$$

$$\in AA_{[i+j-1]} + [Id(A_{[j]}), A_{[i-1]}] \subseteq Id(A_{[i+j-1]}).$$

The proof is completed.

We also note that if i and j are even then $Id(A_{[i]})Id(A_{[j]}) \nsubseteq Id(A_{[i+j-1]})$ in general for associative algebras [61]. Since associative algebras are assosymmetric algebras, we know that if i and j are even then $Id(A_{[i]})Id(A_{[j]}) \nsubseteq Id(A_{[i+j-1]})$ in general for assosymmetric algebras.

3.2 The algebraic classification of nilpotent assosymmetric algebras

The results of this subsection were published in [41 – 43].

Using the classification of all 2-dimensional algebras [62], it is easy to check that all 2-dimensional assosymmetric algebras are associative. The present section presents the algebraic classification of 4-dimensional complex nilpotent assosymmetric algebras.

The variety of assosymmetric algebras is defined by the following identities of right- and left-symmetric:

$$(x,y,z) \;=\; (x,z,y), \qquad (x,y,z) \;=\; (y,x,z),$$

where $(x,y,z) = (xy)z - x(yz)$.

Central extensions are a crucial aspect of our method for classifying assosymmetric nilpotent algebras. The central extensions of Lie and non-Lie algebras have been extensively studied over the years and have been used to classify various types of algebras [36, 63, 64]. Firstly, Skjelbred and Sund devised a method for classifying nilpotent Lie algebras employing central extensions [36]. This method has been utilized to describe all non-Lie central extensions of 4-dimensional Malcev algebras [63], all anticommutative central extensions of 3-dimensional anticommutative algebras [65], and all central extensions of 2-dimensional algebras [66].

3.2.1 Method of classification of nilpotent algebras

We now present an adaptation of the Skjellbred-Sund method for the classification of nilpotent asymmetric algebras. This method has been used to classify various varieties of algebras and has been explained in works such as [63, p. 34], [66]. We give only some important definitions. For more detailed information, the interested reader is referred to these sources. We will also use their notation.

Define an assosymmetric algebra $(A, \cdot)$ over the field of complex numbers $\mathbb{C}$ and let V be a vector space over $\mathbb{C}$. The set of all bilinear maps $\theta: A \times A \longrightarrow V$ satisfying the condition that

$$\theta(xy,z) - \theta(x,yz) = \theta(xz,y) - \theta(x,zy),$$

$$\theta(xy,z)-\theta(x,yz)=\theta(yx,z)-\theta(y,xz).$$

forms a $\mathbb{C}$-linear space, which is denoted as $\mathrm{Z}^2(A,V)$. These maps are referred as *cocycles*. Given a linear map f from A to V, we can define cocycle $\delta f: A\times A\longrightarrow V$ with $\delta f(x,y)=f(xy)$. *Coboundaries* are defined as the elements of the linear subspace $\mathrm{B}^2(A,V)=\{\theta=\delta f\ : f\in \mathrm{H}om(A,V)\}$. The *second cohomology space* $\mathrm{H}^2(A,V)$ is defined to be the quotient space $\mathrm{Z}^2(A,V)/\mathrm{B}^2(A,V)$.

Let $\mathrm{A}ut(A)$ be the automorphism group of the assosymmetric algebra A and let $\phi\in \mathrm{A}ut(A)$. Every $\theta\in \mathrm{Z}^2(A,V)$ defines $\phi\theta(x,y)=\theta(\phi(x),\phi(y))$, with $\phi\theta\in \mathrm{Z}^2(A,V)$. It is easily checked that $\mathrm{A}ut(A)$ acts on $\mathrm{Z}^2(A,V)$, and that $\mathrm{B}^2(A,V)$ is invariant under the action of $\mathrm{A}ut(A)$. So, we have that $\mathrm{A}ut(A)$ acts on $\mathrm{H}^2(A,V)$.

Let A be an assosymmetric algebra of dimension $m<n$ over $\mathbb{C}$, V a $\mathbb{C}$-vector space of dimension $n-m$ and θ a cocycle, and consider the direct sum $A_\theta=A\oplus V$ with the bilinear product "$[-,-]_{A_\theta}$" defined by $[x+x',y+y']_{A_\theta}=xy+\theta(x,y)$ for all $x,y\in A, x',y'\in V$. It is straightforward that A_θ is an assosymmetric algebra if and only if $\theta\in \mathrm{Z}^2(A,V)$; it is called an $(n-m)$- *dimensional central extension* of A by V.

We also call the set $\mathrm{A}nn(\theta)=\{x\in A: \theta(x,A)+\theta(A,x)=0\}$ the *annihilator* of θ. We recall that the *annihilator* of an algebra A is defined as the ideal $\mathrm{A}nn(A)=\{x\in A: xA+Ax=0\}$. Observe that $\mathrm{A}nn(A_\theta)=(\mathrm{A}nn(\theta)\cap \mathrm{A}nn(A))\oplus V$.

Definition 3.2.1 [63, p. 35] *Let A be an algebra and I be a subspace of* $\mathrm{A}nn(A)$. *If* $A=A_0\oplus I$ *then I is called an annihilator component of A.*

Definition 3.2.2 [63, p. 35] *A central extension of an algebra A without annihilator component is called a non-split central extension.*

The following result is fundamental for the classification method. For the proof, we refer the reader to Lemma 5 in [63, p. 35].

Lemma 3.2.3 *Let A be an* n*-dimensional assosymmetric algebra such that* $dim\ Ann(A)=m\neq 0$. *Then there exists, up to isomorphism, a unique* $(n-m)$*-dimensional assosymmetric algebra* A' *and a bilinear map* $\theta\in Z^2(A,V)$ *with* $Ann(A)\cap Ann(\theta)=0$, *where V is a vector space of dimension m, such that* $A\cong A'_\theta$ *and* $A/Ann(A)\cong A'$.

Now, we seek a condition on the cocycles to know when two $(n-m)$-central extensions are isomorphic. Let us fix a basis $e_1,\dots,e_s$ of V, and $\theta\in \mathrm{Z}^2(A,V)$. Then θ can be uniquely written as $\theta(x,y)=\sum_{i=1}^{s}\theta_i(x,y)e_i$, where $\theta_i\in \mathrm{Z}^2(A,\mathbb{C})$. It holds that $\theta\in \mathrm{B}^2(A,V)$ if and only if all $\theta_i\in \mathrm{B}^2(A,\mathbb{C})$, and it also holds that $\mathrm{A}nn(\theta)=\mathrm{A}nn(\theta_1)\cap \mathrm{A}nn(\theta_2)\dots\cap \mathrm{A}nn(\theta_s)$. Furthermore, if $\mathrm{A}nn(\theta)\cap \mathrm{A}nn(A)=0$, then A_θ has an annihilator component if and only if $[\theta_1],[\theta_2],\dots,[\theta_s]$ are linearly dependent in $\mathrm{H}^2(A,\mathbb{C})$ (see [63, Lemma 13]).

Recall that, given a finite-dimensional vector space V over $\mathbb{C}$, the *Grassmannian* $\mathrm{G}_k(V)$ is the set of all k-dimensional linear subspaces of V. Let $\mathrm{G}_s(\mathrm{H}^2(A,\mathbb{C}))$ be the Grassmannian of subspaces of dimension s in $\mathrm{H}^2(A,\mathbb{C})$. For $\mathrm{W}=\langle[\theta_1],[\theta_2],\dots,[\theta_s]\rangle\in \mathrm{G}_s(\mathrm{H}^2(A,\mathbb{C}))$ and $\phi\in \mathrm{A}ut(A)$, define $\phi\mathrm{W}=\langle[\phi\theta_1],[\phi\theta_2],\dots,[\phi\theta_s]\rangle$. It holds that $\phi\mathrm{W}\in \mathrm{G}_s(\mathrm{H}^2(A,\mathbb{C}))$, and this induces an action of $\mathrm{A}ut(A)$ on $\mathrm{G}_s(\mathrm{H}^2(A,\mathbb{C}))$. We denote the orbit of $\mathrm{W} \in \mathrm{G}_s(\mathrm{H}^2(A,\mathbb{C}))$ under this action by $\mathrm{O}rb(\mathrm{W})$. Let

$$\mathrm{W}_1 = \langle[\theta_1],[\theta_2],\dots,[\theta_s]\rangle, \mathrm{W}_2 = \langle[\vartheta_1],[\vartheta_2],\dots,[\vartheta_s]\rangle \in \mathrm{G}_s(\mathrm{H}^2(A,\mathbb{C})).$$

Similarly to Lemma 15 in [63, p. 41], in case $\mathrm{W}_1 = \mathrm{W}_2$, it holds that

$$\bigcap_{i=1}^{s} \mathrm{A}nn(\theta_i) \cap \mathrm{A}nn(A) = \bigcap_{i=1}^{s} \mathrm{A}nn(\vartheta_i) \cap \mathrm{A}nn(A),$$

and therefore the set

$$\mathrm{T}_s(A) = \left\{\mathrm{W} = \langle[\theta_1],[\theta_2],\dots,[\theta_s]\rangle \in \mathrm{G}_s(\mathrm{H}^2(A,\mathbb{C})) : \bigcap_{i=1}^{s} \mathrm{A}nn(\theta_i) \cap \mathrm{A}nn(A) = 0\right\}$$

is well defined, and it is also stable under the action of $\mathrm{A}ut(A)$ (see Lemma 16 in [63, p. 41]). Now, let V be an s-dimensional linear space, and let us denote by $\mathrm{E}(A,V)$ the set of all non-split s-dimensional central extensions of A by V. We can write

$$\mathrm{E}(A,V) = \left\{A_\theta : \theta(x,y) = \sum_{i=1}^{s} \theta_i(x,y)e_i \quad \text{and} \quad \langle[\theta_1],[\theta_2],\dots,[\theta_s]\rangle \in \mathrm{T}_s(A)\right\}.$$

Finally, we have main lemma, which can be proved as Lemma 17 in [63].

Lemma 3.2.4 [63, p. 41] *Let $A_\theta, A_\vartheta \in E(A,V)$. Suppose $\theta(x,y) = \sum_{i=1}^{s} \theta_i(x,y)e_i$ and $\vartheta(x,y) = \sum_{i=1}^{s} \vartheta_i(x,y)e_i$. Then the assosymmetric algebras A_θ and A_ϑ are isomorphic if and only if*

$$\mathrm{O}rb\langle[\theta_1],[\theta_2],\dots,[\theta_s]\rangle = \mathrm{O}rb\langle[\vartheta_1],[\vartheta_2],\dots,[\vartheta_s]\rangle.$$

Then, it exists a bijective correspondence between the set of $\mathrm{A}ut(A)$-orbits on $\mathrm{T}_s(A)$ and the set of isomorphism classes of $\mathrm{E}(A,V)$. Consequently, we have a procedure that allows us, given an assosymmetric algebra A' of dimension $n-s$, to construct all non-split central extensions of A'.

Let A' be an assosymmetric algebra of dimension $n-s$. Then:

1. Compute base for $Z^2(A',\mathbb{C})$;
2. Compute base for $B^2(A',\mathbb{C})$ and $H^2(A',\mathbb{C})$;
3. Compute $Aut(A')$;
4. Compute base for $Ann(A')$ and $Ann(A') \cap Ann(\theta)$;
5. Compute $\mathrm{A}ut(A')$-orbits on $T_m(A')$;
6. Construct a new finite-dimensional nilpotent assosymmetric algebra associated with a representative of each orbit.

Let A be an assosymmetric algebra and fix a basis $e_1, e_2, \dots, e_n$. We define the bilinear form $\Delta_{ij}: A \times A \longrightarrow \mathbb{C}$ by $\Delta_{ij}(e_l, e_m) = \delta_{il}\delta_{jm}$. Then the set $\{\Delta_{ij}: 1 \le i, j \le n\}$ is a basis for the linear space of the bilinear forms on A, and in particular, every $\theta \in \mathrm{Z}^2(A, V)$ can be uniquely written as $\theta = \sum_{1 \le i,j \le n} c_{ij}\Delta_{ij}$, where $c_{ij} \in \mathbb{C}$.

We now describe algorithms to handle steps from 1 to 4. The remaining two steps are worked out by hand.

Let A' be an algebra with basis $\{e_i: i = 1,2, \dots, n\}$. We use the following notations: $\Delta_{i,j}$ is the bilinear form $\Delta_{i,j}: A' \times A' \longrightarrow \mathbb{C}$ such that

$$\Delta_{i,j}(e_l, e_k) = \delta_{il}\delta_{jk}. \tag{34}$$

The set $\{\Delta_{i,j}: 1 \le i, j \le n\}$ is the basis of $Z^2(A', \mathbb{C})$. Every $\theta \in Z^2(A', \mathbb{C})$ can be uniquely written as $\theta = \sum_{1 \le i,j \le n}^{n} \lambda_{i,j}\Delta_{i,j}$ where $\lambda_{i,j} \in \mathbb{C}$.

Now, we give algorithms to compute the above-mentioned steps. The first algorithm shows how to compute $Z^2(A', \mathbb{C})$ given the dimension, the product rule, and the polynomial identities. It amounts to defining the symbolic equations and calling the symbolic solver from the relevant programming language.

Algorithm 1: Algorithm to compute the basis for the $Z^2(A', \mathbb{C})$

Input: Dimension of your algebra: n, Bilinear product rule: $\mathrm{pr}(\cdot, \cdot)$,
Identities: $\{\mathrm{Iden}_1, \dots, \mathrm{Iden}_m\}$.
Define a symbolic basis for algebra A': $\{e_1, \dots, e_n\}$
Define symbolic cocycles, a bilinear function: $\theta(\cdot, \cdot) = \sum_{i,j}^{n} \lambda_{i,j}\Delta_{i,j}(\cdot, \cdot)$
Let k be the number of variables used in the Identities.
for $(i_1, \dots, i_k) \in \{1, \dots, n\}^k$ **do**
    for $j = 1 : m$ **do**
        Define an $eq[i_1, \dots, i_k, j]$ that is obtained by applying θ to Iden_j when $e[i_1], \dots, e[i_k]$ are substituted.
    end
end
Use the built-in 'solve' function to solve the system $\{eq\}$.
Output: Basis for $Z^2(A', \mathbb{C})$: Z2A

Figure 1 – Computation the basis for the $Z^2(A', \mathbb{C})$

The next algorithm uses outcomes of Algorithm 1 (Figure 1) together with the same inputs. In this case we aim to compute bases for $B^2(A', \mathbb{C})$ and $H^2(A', \mathbb{C})$. It does not require any tricks to obtain a basis for $B^2(A', \mathbb{C})$ but simply write them down manually from the given polynomial identities. In terms of coding, this means asking the programming language to read the coefficients of polynomial expressions. As for the second part, we recall that $B^2(A', \mathbb{C}) \subset Z^2(A', \mathbb{C})$ and $H^2(A', V) = Z^2_{\mathfrak{L}}(A', V)/$

$B^2(A', V)$. Thus, the problem of finding a basis for $H^2(A', \mathbb{C})$ is equivalent to completing the basis of $Z^2(A', \mathbb{C})$ given the basis of $B^2(A', \mathbb{C})$.

Algorithm 2: Algorithm to compute the bases for $B^2(A', \mathbb{C})$ and $H^2(A', \mathbb{C})$

Input: Dimension of your algebra: n, Bilinear product rule: $\mathrm{pr}(\cdot, \cdot)$,
Identities: $\{\mathrm{Iden}_1, \ldots, \mathrm{Iden}_m\}$, Basis for $Z^2(A', \mathbb{C}) : \{z_1, \ldots, z_k\}$

From $\mathrm{pr}(\cdot, \cdot)$ obtain a basis for $B^2(A') : B2A = \{b_1, \ldots, b_s\}$

Define empty set $H2A := \{\}$;

for $i = 1 : k$ **do**

  if $z_i \notin span(B2A)$ **then**

    Add z_i to $H2A$;

    Add z_i to $B2A$;

  end

end

Output: Basis for $B^2(A', \mathbb{C})$ and $H^2(A', \mathbb{C})$: B2A, H2A

Figure 2 – Computation the bases for $B^2(A', \mathbb{C})$ and $H^2(A', \mathbb{C})$

Computing the $Aut(A')$, Algorithm 3 (Figure 3), is one of the main steps in the above-described method and the one with a large computational cost. We may represent an automorphism with an $n \times n$ invertible square matrix that respects the bilinear product rule. This requires defining a symbolic matrix and defining a system of symbolic equations and finally calling the solve function.

Algorithm 3: Algorithm to find the automorphism group

Input: Dimension of your algebra: n, Bilinear product rule: $\mathrm{pr}(\cdot, \cdot)$

Define matrix $\mathrm{GAut}_{n\times n}$; Define homomorphism function
$F[\{x, y\}] = pr[F[x], F[y]] - F[pr[x, y]]$;

Define mapping of basis by $F[e_i] = \sum_{j=1}^{n} \lambda_{i,j} e_j$;

for $i,j;\ n$ **do**

  Substitute basis to $F[\{e_i, e_j\}]$ and define by Eq

end

Use "solve" to find is a set of solutions λ_{ij} and defined by Solution;

Obtained solutions substitute to matrix GAut, that is

for $i = 1; Length[Solutions]$ **do**

  $Mat[i] = GAut/.Solution[[i]]$

end

Define set Automorphism={};

if $Det[Mat[i]]! = 0$ **then**

  Add to Automorphism

end

Output: Matrix forms of Automorphism group: Automorhism

Figure 3 – Finding the automorphism group

The next three algorithms are allocated for step 4 to compute annihilators. The Algorithm 4 (Figure 4) computes the action of the automorphism group on $H^2(A', \mathbb{C})$ and uses outcomes of Algorithm 2 (Figure 2) and Algorithm 3 (Figure 3). Action of automorphism group defined by $\phi^T * M * \phi$ where $\phi \in Aut(A')$ and M is matrix form of $H^2(A', \mathbb{C})$.

Algorithm 4: Algorithm of action of the automorphism group on $H^2(A', \mathbb{C})$

Input: Automorphism group of algebra, basis of $H^2(A', \mathbb{C})$.
for $i = 1; Length[Automorphism]$ **do**
  $ActAut[i] =$
  $Transpose[Automorphism[[i]]].MatrixFormH2.Automorphism[[i]]$
end
Output: Action of automorphism group on $H^2(A', \mathbb{C})$: ActAut[i]

Figure 4 – Action of the automorphism group on $H^2(A', \mathbb{C})$

Next algorithm uses Algorithm 5 (Figure 5) to compute bases for $Ann(A')$. Again one needs to define the system of polynomial equations and call the solver.

Algorithm 5: Finding basis of annihilator.

Input: Multiplication of basis elements
Define linear combinations of basis elements of algebra by SumElemOFAnn;
Define empty set by ProdSumElemOFAnn;
for $i = 1; dim$ **do**
  pr[SumElemOFAnn, e[i]] == 0 add to ProdSumElemOFAnn;
  pr[e[i], SumElemOFAnn] == 0 add to ProdSumElemOFAnn;
end
Solve ProdSumElemOFAnn after obtaining solution put to SumElemOFAnn;
Output: Basis of Annihilator: SpanAnn.

Figure 5 – Finding basis of annihilator

Finally, the last algorithm below (Figure 6) uses outcome of Algorithm 5 (Figure 5) and gives conditions of $Ann(A') \cap Ann(\theta) = 0$:

Algorithm 6: Intersection condition of $Ann(A')$ and $Ann(\theta)$:

Input: A' is n dimensional algebra, basis of $Ann(A')$ and basis of $H^2(A', \mathbb{C})$

Define the following :

Linear combinations of basis elements of $Ann(A')$ by SpanAnn1.

Define $\Delta_{i,j}$

OpenBraket[x_, y_] := x[y];

ConditionOfAnnAndTheta = {};

for *j;n+1* **do**

  for *i;Length[H2]* **do**

    Sum[α_iOpenBraket[H2[[i]]],{SpanAnn1[[1]], e[j]}]==0 add to ConditionOfAnnAndTheta;

    Sum[α_iOpenBraket[H2[[i]]],{e[j],SpanAnn1[[1]]}]==0 add to ConditionOfAnnAndTheta;

  end

end

Output: Condition of $Ann(A') \cap Ann(\theta) = 0$: ConditionOfAnnAndTheta

Figure 6 – Intersection of $Ann(A')$ and $Ann(\theta)$

3.2.2 The central extensions of low dimensional nilpotent assosymmetric algebras.

We distinguish two main classes of assosymmetric algebras: the "pure" and the "non-pure" ones. By the non-pure ones, we mean those satisfying the identities $(xy)z = 0$ and $x(yz) = 0$; the pure ones are the rest.

These "trivial" algebras can be considered in many varieties of algebras defined by polynomial identities of degree 3 (associative, Leibniz, Zinbiel, etc.), and they can be expressed as central extensions of suitable algebras with zero product. Those with dimension 4 are already classified: the list of the non-anticommutative ones can be found in [67], and there is only one nilpotent and anticommutative algebra.

Theorem 3.2.5 *Let A be a nonzero* 4*-dimensional complex nilpotent "pure" assosymmetric algebra. Then, A is isomorphic to one of the algebras listed in Table A.1 in Appendix A.*

Remark 3.2.6 *Let A be a* 4*-dimensional nilpotent non-associative assosymmetric algebra. Then A is isomorphic to one algebra from the following list*

$$A^4_{01}, A^4_{02}(\alpha \neq 1), A^4_{03}, A^4_{04}(\alpha \neq 1), A^4_{06}(\alpha \neq 1), A^4_{07}, A^4_{08}, A^4_{09}, A^4_{10},$$

$$A^4_{11}, A^4_{12}, A^4_{13}, A^4_{14}, A^4_{15}, A^4_{16}, A^4_{17}, A^4_{18}(\alpha \neq 1), A^4_{19}, A^4_{20}.$$

Proof of Theorem 3.2.5. There are no nontrivial 1-dimensional nilpotent assosymmetric algebras, and there is only one nontrivial 2-dimensional nilpotent assosymmetric algebra (namely, the non-split central extension of the 1-dimensional algebra with zero product):

$$A_{01}^{2*} \quad : \quad (\mathfrak{N}_1)_{2,1} \quad : \quad e_1e_1 = e_2.$$

From this algebra, we construct the 3-dimensional nilpotent assosymmetric algebra $A_{01}^{3*} = A_{01}^{2*} \oplus \mathbb{C}e_3$. Also, the reference [66, p. 10] gives the description of all central extensions of A_{01}^{2*} and $\mathfrak{N}_2$. Choosing the assosymmetric algebras between them, we have the classification of all non-split 3-dimensional nilpotent assosymmetric algebras:

$$\begin{array}{llllll}
A_{02}^{3*} & : & (\mathfrak{N}_2)_{3,1} & : & e_1e_1 = e_3, & e_2e_2 = e_3; \\
A_{03}^{3*} & : & (\mathfrak{N}_2)_{3,2} & : & e_1e_2 = e_3, & e_2e_1 = -e_3; \\
A_{04}^{3*}(\alpha) & : & (\mathfrak{N}_2)_{3,3} & : & e_1e_1 = \alpha e_3, & e_2e_1 = e_3, \quad e_2e_2 = e_3; \\
A_{01}^{3} & : & (A_{01}^{2*})_{3,1} & : & e_1e_1 = e_2, & e_2e_1 = e_3; \\
A_{02}^{3}(\alpha) & : & (A_{01}^{2*})_{3,2} & : & e_1e_1 = e_2, & e_1e_2 = e_3, \quad e_2e_1 = \alpha e_3.
\end{array}$$

Now we consider 1-dimensional central extensions of 3-dimensional nilpotent assosymmetric algebras. In the following table, we give the description of the second cohomology space of 3-dimensional nilpotent assosymmetric algebras.

Table 1 – the cohomology space of 3-dimensional nilpotent assosymmetric algebras

A	$\mathrm{Z}^2(A)$	$\mathrm{B}^2(A)$	$\mathrm{H}^2(A)$
A_{01}^{2*}	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{21}, \Delta_{31}, \Delta_{33} \rangle$	$\langle \Delta_{11} \rangle$	$\langle [\Delta_{12}], [\Delta_{13}], [\Delta_{21}], [\Delta_{31}], [\Delta_{33}] \rangle$
A_{02}^{3*}	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22} \rangle$	$\langle \Delta_{11} + \Delta_{22} \rangle$	$\langle [\Delta_{12}], [\Delta_{21}], [\Delta_{22}] \rangle$
A_{03}^{3*}	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22} \rangle$	$\langle \Delta_{12} - \Delta_{21} \rangle$	$\langle [\Delta_{11}], [\Delta_{21}], [\Delta_{22}] \rangle$
$A_{04}^{3*}(\alpha \neq 1)$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22} \rangle$	$\langle \alpha\Delta_{11} + \Delta_{21} + \Delta_{22} \rangle$	$\langle [\Delta_{12}], [\Delta_{21}], [\Delta_{22}] \rangle$
$A_{04}^{3*}(1)$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{31} + \Delta_{23}, \Delta_{22}, \Delta_{13} - \Delta_{32} + \Delta_{23} \rangle$	$\langle \Delta_{11} + \Delta_{21} + \Delta_{22} \rangle$	$\langle [\Delta_{12}], [\Delta_{21}], [\Delta_{31}] + [\Delta_{23}], [\Delta_{22}], [\Delta_{13}] - [\Delta_{32}] + [\Delta_{23}] \rangle$
A_{01}^{3}	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} - \Delta_{22} - 2\Delta_{31}$	$\langle \Delta_{11}, \Delta_{21} \rangle$	$\langle [\Delta_{12}], [\Delta_{13}] - [\Delta_{22}] - 2[\Delta_{31}] \rangle$
$A_{02}^{3}(\alpha)$	$\Delta_{11}, \Delta_{12}, \Delta_{21},$ $\langle (\alpha-2)\Delta_{13} + (1-2\alpha)\Delta_{31} + \rangle$ $(\alpha - \alpha^2 - 1)\Delta_{22}$	$\langle \Delta_{11}, \Delta_{12} + \alpha\Delta_{21} \rangle$	$[\Delta_{21}],$ $\langle (\alpha-2)[\Delta_{13}] + (1-2\alpha)[\Delta_{31}] + \rangle$ $(\alpha - \alpha^2 - 1)[\Delta_{22}]$

Remark 3.2.7 *From the description of the cocycles of the algebras A_{02}^{3*}, A_{03}^{3*} and $A_{04}^{3*}(\alpha)_{\alpha\neq 1}$, it follows that the 1-dimensional central extensions of these algebras are 2-dimensional central extensions of 2-dimensional nilpotent assosymmetric algebras.* Thanks to [66, p. 18-22] we have the description of all non-split 2-dimensional central extensions of 2-dimensional nilpotent assosymmetric algebras:

$$A_{03}^{4} \quad : \quad (A_{01}^{2*})_{4,1} \quad : \quad e_1e_1 = e_2, \quad e_1e_2 = e_4, \quad e_2e_1 = e_3.$$

Then, in the following we study the central extensions of the other algebras.

1) Central extensions of A_{01}^{3*}. Since the second cohomology spaces and automorphism groups of A_{01}^{3*} and $\mathcal{N}_{01}^{3*}$ (from [11]) coincide, these algebras have the same central extensions. Therefore, from [11, p. 19] we have all the new 4-dimensional nilpotent assosymmetric algebras constructed from A_{01}^{3*}:

$$A_{04}^{4}(\alpha),\qquad A_{05}^{4},\qquad A_{06}^{4}(\alpha)_{\alpha\neq 0},\qquad A_{07}^{4},\qquad A_{08}^{4},\qquad A_{09}^{4}.$$

The multiplication tables of these algebras can be found in Appendix A.

2) Central extensions of $A_{04}^{3*}(1)$. Let us use the following notations:

$$\nabla_1=[\Delta_{12}],\ \nabla_2=[\Delta_{21}],\nabla_3=[\Delta_{22}],$$

$$\nabla_4=[\Delta_{13}]-[\Delta_{32}]+[\Delta_{23}],\nabla_5=[\Delta_{31}]+[\Delta_{23}].$$

The automorphism group of $A_{04}^{3*}(1)$ consists of invertible matrices of the form

$$\phi=\begin{pmatrix} x & y & 0\\ -y & x-y & 0\\ z & u & x^2-xy+y^2\end{pmatrix}.$$

Since

$$\phi^T\begin{pmatrix} 0 & \alpha_1 & \alpha_4\\ \alpha_2 & \alpha_3 & \alpha_4+\alpha_5\\ \alpha_5 & -\alpha_4 & 0\end{pmatrix}\phi=\begin{pmatrix} \alpha^* & \alpha_1^* & \alpha_4^*\\ \alpha^*+\alpha_2^* & \alpha^*+\alpha_3^* & \alpha_4^*+\alpha_5^*\\ \alpha_5^* & -\alpha_4^* & 0\end{pmatrix},$$

we have that the action of $Aut\left(A_{04}^{3*}(1)\right)$ on the subspace $\left\langle\sum_{i=1}^{5}\alpha_i\nabla_i\right\rangle$ is given by $\left\langle\sum_{i=1}^{5}\alpha_i^*\nabla_i\right\rangle$, where

$$\begin{aligned}
\alpha_1^* &= x(x-y)\alpha_1-y^2\alpha_2-y(x-y)\alpha_3+(u-z)(x-y)\alpha_4-y(u-z)\alpha_5;\\
\alpha_2^* &= y(x-y)\alpha_1+x^2\alpha_2-xy\alpha_3+u(y\alpha_4+x\alpha_5);\\
\alpha_3^* &= y(2x-y)\alpha_1+y(2x-y)\alpha_2+x(x-2y)\alpha_3+(uy-xz)\alpha_4+\\
&\quad +\left(ux-z(x-y)\right)\alpha_5;\\
\alpha_4^* &= (x^2-xy+y^2)\left((x-y)\alpha_4-y\alpha_5\right);\\
\alpha_5^* &= (x^2-xy+y^2)(y\alpha_4+x\alpha_5).
\end{aligned}$$

The element $\alpha_1\nabla_1+\alpha_2\nabla_2+\alpha_3\nabla_3$ gives a central extension of a 2-dimensional algebra. From here, we have the following new cases:

1. $\alpha_5=0,\alpha_4\neq 0$. Then

(a) If $\alpha_2 = 0$, then choosing $x = \frac{1}{\sqrt[3]{\alpha_4}}, y = 0, u = \frac{x(\alpha_3-\alpha_1)}{\alpha_4}, z = \frac{x\alpha_3}{\alpha_4}$, we have the representative $\langle \nabla_4 \rangle$.

(b) If $\alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_4}, y = 0, u = \frac{x(\alpha_3-\alpha_1)}{\alpha_4}, z = \frac{x\alpha_3}{\alpha_4}$, we have the representative$\langle \nabla_2 + \nabla_4 \rangle$.

2. $\alpha_5 \neq 0$ and $\alpha_4^2 + \alpha_4\alpha_5 + \alpha_5^2 \neq 0$. Then by chossing $x = -\frac{\alpha_4}{\alpha_5}, y = 1, u = 0, z = 0$ we have the case (1).

3. $\alpha_5 \neq 0$ and $\alpha_4^2 + \alpha_4\alpha_5 + \alpha_5^2 = 0$. Then $\alpha_5 = q\alpha_4$, where $q = \frac{-1\pm\sqrt{3}i}{2}$. Now we have following cases.

(a) If $\alpha_1 + \alpha_2 - \alpha_3(1+q^2) \neq 0$, then by choosing

$$x = \frac{\alpha_1 + \alpha_2 - \alpha_3(1+q^2)}{\alpha_5}, y = 0, u = -\frac{x\alpha_2}{\alpha_5}, z = \frac{x(q\alpha_1 - \alpha_2)}{\alpha_5},$$

we have the representative $\langle \nabla_3 + \nabla_4 + q\nabla_5 \rangle$.

(b) If $\alpha_1 + \alpha_2 - \alpha_3(1+q^2) = 0$, then by choosing

$$x = \frac{1}{\sqrt[3]{\alpha_4}}, y = 0, u = -\frac{x\alpha_2}{\alpha_5}, z = \frac{x(q\alpha_1 - \alpha_2)}{\alpha_5},$$

we have the representative $\langle \nabla_4 + q\nabla_5 \rangle$.

Now, we have the following new algebras constructed from $A_{04}^{3*}(1)$:

A_{10}^4: $e_1e_1 = e_3$ $\quad e_1e_3 = e_4$ $\quad e_2e_1 = e_3$ $\quad e_2e_2 = e_3$
$e_2e_3 = e_4$ $\quad e_3e_2 = -e_4$.

A_{11}^4: $e_1e_1 = e_3$ $\quad e_1e_3 = e_4$ $\quad e_2e_1 = e_3 + e_4$ $\quad e_2e_2 = e_3$
$e_2e_3 = e_4$ $\quad e_3e_2 = -e_4$;

A_{12}^4: $e_1e_1 = e_3$ $\quad e_1e_3 = e_4$ $\quad e_2e_1 = e_3$ $\quad e_2e_2 = e_3 + e_4$
$e_2e_3 = \frac{1+\sqrt{3}i}{2}e_4$ $\quad e_3e_1 = \frac{-1+\sqrt{3}i}{2}e_4$ $\quad e_3e_2 = -e_4$;

A_{13}^4: $e_1e_1 = e_3$ $\quad e_1e_3 = e_4$ $\quad e_2e_1 = e_3$ $\quad e_2e_2 = e_3$
$e_2e_3 = \frac{1+\sqrt{3}i}{2}e_4$ $\quad e_3e_1 = \frac{-1+\sqrt{3}i}{2}e_4$ $\quad e_3e_2 = -e_4$;

A_{14}^4: $e_1e_1 = e_3$ $\quad e_1e_3 = e_4$ $\quad e_2e_1 = e_3$ $\quad e_2e_2 = e_3 + e_4$
$e_2e_3 = \frac{1-\sqrt{3}i}{2}e_4$ $\quad e_3e_1 = \frac{-1-\sqrt{3}i}{2}e_4$ $\quad e_3e_2 = -e_4$;

A_{15}^4: $e_1e_1 = e_3$ $\quad e_1e_3 = e_4$ $\quad e_2e_1 = e_3$ $\quad e_2e_2 = e_3$
$e_2e_3 = \frac{1-\sqrt{3}i}{2}e_4$ $\quad e_3e_1 = \frac{-1-\sqrt{3}i}{2}e_4$ $\quad e_3e_2 = -e_4$.

3) Central extensions of A_{01}^3. Let us use the following notations:

$$\nabla_1 = [\Delta_{12}], \nabla_2 = [\Delta_{13}] - [\Delta_{22}] - 2[\Delta_{31}].$$

The automorphism group of A_{01}^3 consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 \\ y & x^2 & 0 \\ z & xy & x^3 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ 0 & -\alpha_2 & 0 \\ -2\alpha_2 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha_1^* & \alpha_2^* \\ \alpha^{**} & -\alpha_2^* & 0 \\ -2\alpha_2^* & 0 & 0 \end{pmatrix},$$

we have that the action of $Aut(A_{01}^3)$ on the subspace $\langle \sum_{i=1}^2 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^2 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = x^3 \alpha_1; \quad \alpha_2^* = x^4 \alpha_2.$$

It is straightforward that the elements $\alpha_1 \nabla_1$ lead to central extensions of 2-dimensional algebras. The new cases are following:

1. $\alpha_1 \neq 0, \alpha_2 \neq 0$. Choosing $x = \frac{\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_1 + \nabla_2 \rangle$.
2. $\alpha_1 = 0, \alpha_2 \neq 0$. Choosing $x = \frac{1}{\sqrt[4]{\alpha_2}}$, we have the representative $\langle \nabla_2 \rangle$.

We have the following new algebras constructed from A_{01}^3:

A_{16}^4: $e_1e_1 = e_2$ $\quad e_2e_1 = e_3$ $\quad e_1e_2 = e_4$ $\quad e_1e_3 = e_4$
$e_2e_2 = -e_4$ $\quad e_3e_1 = -2e_4$;

A_{17}^4: $e_1e_1 = e_2$ $\quad e_2e_1 = e_3$ $\quad e_1e_3 = e_4$ $\quad e_2e_2 = -e_4$
$e_3e_1 = -2e_4$.

4) Central extensions of $A_{02}^3(\alpha)$. Let us use the following notations:

$$\nabla_1 = [\Delta_{21}], \nabla_2 = (\alpha - 2)[\Delta_{13}] + (\alpha - \alpha^2 - 1)[\Delta_{22}] + (1 - 2\alpha)[\Delta_{31}].$$

The automorphism group of $A_{02}^3(\alpha)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 \\ y & x^2 & 0 \\ z & (\alpha+1)xy & x^3 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & (\alpha-2)\alpha_2 \\ \alpha_1 & (\alpha-\alpha^2-1)\alpha_2 & 0 \\ (1-2\alpha)\alpha_2 & 0 & 0 \end{pmatrix} \phi =$$

$$\begin{pmatrix} \alpha^{**} & \alpha^{*} & (\alpha-2)\alpha_2^* \\ \alpha\alpha^* + \alpha_1^* & (\alpha-\alpha^2-1)\alpha_2^* & 0 \\ (1-2\alpha)\alpha_2^* & 0 & 0 \end{pmatrix},$$

we have that the action of $Aut\big(A_{02}^3(\alpha)\big)$ on the subspace $\langle \sum_{i=1}^{2} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{2} \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = x^3\alpha_1 - 3(\alpha+1)\alpha x^2 y \alpha_2;\ \alpha_2^* = x^4\alpha_2.$$

The element $\alpha_1\nabla_1$ gives a central extension of a 2-dimensional algebra, then we will consider only cases with $\alpha_2 \neq 0$. We find the following new cases:

1. $\alpha \neq 0, -1$, then choosing $x = -\frac{1}{\sqrt[4]{\alpha_2}}$ and $y = \frac{x\alpha_1}{3(\alpha+1)\alpha\alpha_2}$, we have the representative $\langle -\nabla_2 \rangle$.
2. $\alpha = 0$ or $\alpha = -1$ then:
   (a) if $\alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_1 + \nabla_2 \rangle$.
   (b) if $\alpha_1 = 0$, then choosing $x = \frac{1}{\sqrt[4]{\alpha_2}}$, we have the representative $\langle \nabla_2 \rangle$.

Now we have all the new 4-dimensional nilpotent assosymmetric algebras constructed from $A_{02}^3(\alpha)$: $A_{18}^4(\alpha)$, A_{19}^4, A_{20}^4 (see Table A.1 in Appendix A).

Summarizing above results the Theorem 3.2.6 is proved.

Another main result of the present section is the following theorem:

Theorem 3.2.8 *Let A be a* 5- *or* 6-*dimensional complex one-generated nilpotent assosymmetric algebra, then A is isomorphic to an algebra from the Table A.3 or Table A.5 in Appendix A.*

From Theorem 3.2.5 we have a description of all 2-, 3- and 4-dimensional one-generated nilpotent assosymmetric algebras:

Table 2 – 2-, 3- and 4-dimensional one-generated nilpotent assosymmetric algebras

A_{01}^2	$e_1e_1 = e_2$		
A_{01}^3	$e_1e_1 = e_3$	$e_2e_1 = e_3$	
$A_{02}^3(\alpha)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_2e_1 = \alpha e_3$
A_{01}^4	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_3$
A_{02}^4	$e_1e_1 = e_2$ $e_2e_1 = e_3$	$e_1e_2 = e_4$ $e_2e_2 = -e_4$	$e_1e_3 = e_4$ $e_3e_1 = -2e_4$
A_{03}^4	$e_1e_1 = e_2$ $e_2e_2 = -e_4$	$e_1e_3 = e_4$ $e_3e_1 = -2e_4$	$e_2e_1 = e_3$
$A_{04}^4(\alpha)$	$e_1e_1 = e_2$ $e_2e_1 = \alpha e_3$	$e_1e_2 = e_3$ $e_2e_2 = (\alpha^2 - \alpha + 1)e_4$	$e_1e_3 = (2-\alpha)e_4$ $e_3e_1 = (2\alpha - 1)e_4$
A_{05}^4	$e_1e_1 = e_2$ $e_2e_1 = e_4$	$e_1e_2 = e_3$ $e_2e_2 = -e_4$	$e_1e_3 = -2e_4$ $e_3e_1 = e_4$
A_{06}^4	$e_1e_1 = e_2$ $e_2e_1 = -e_3 + e_4$	$e_1e_2 = e_3$ $e_2e_2 = -3e_4$	$e_1e_3 = -3e_4$ $e_3e_1 = 3e_4$

Proof Theorem 3.2.8. We consider 2-dimensional central extensions of 3-dimensional one-generated algebras.The second cohomology spaces of algebras $\boldsymbol{A_{01}^3}, \boldsymbol{A_{02}^3(\alpha)}$ given in [43]. Therefore, 2-dimensional central extensions of these algebras gives the following two algebras:

$$\begin{array}{llll} A_{01}^5: & e_1e_1 = e_2 & e_1e_2 = e_4 & e_1e_3 = e_5 \\ & e_2e_1 = e_3 & e_2e_2 = -e_5 & e_3e_1 = -2e_5; \\ A_{02}^5(\alpha): & e_1e_1 = e_2 & e_1e_2 = e_3 & e_1e_3 = (\alpha - 2)e_5 \\ & e_2e_1 = \alpha e_3 + e_4 & e_2e_2 = (\alpha - \alpha^2 - 1)e_5 & e_3e_1 = (1 - 2\alpha)e_5. \end{array}$$

Remark 3.2.9 *Extensions of the algebras A_{02}^4, A_{03}^4, $A_{04}^4(\alpha)_{\alpha\neq 1}$, A_{05}^4 and A_{06}^4 give algebras with 2-dimensional annihilator. Then, in the following subsections we study the central extensions of the other algebras.*

All multiplication tables of 4-dimensional one-generated nilpotent assosymmetric algebras is given in Table A.1 (see, Appendix A). All relevant details about coboundaries, cocycles, and second cohomology spaces for five-dimensional one-generated nilpotent assosymmetric algebras were obtained using the code specified in [41], and can be found in Table A.2 (see, Appendix A).

1) Central extensions of $\boldsymbol{A_{01}^4}$. Let us use the following notations:

$$\nabla_1 = [\Delta_{13}] + [\Delta_{41}], \nabla_2 = [\Delta_{14}] - [\Delta_{31}] - [\Delta_{41}], \nabla_3 = [\Delta_{22}] + 2[\Delta_{31}] + [\Delta_{41}].$$

The automorphism group of A_{01}^4 consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ y & x^2 & 0 & 0 \\ z & xy & x^3 & 0 \\ t & xy & 0 & x^3 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_1 & \alpha_2 \\ 0 & \alpha_3 & 0 & 0 \\ -\alpha_2 + 2\alpha_3 & 0 & 0 & 0 \\ \alpha_1 - \alpha_2 + \alpha_3 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha_1^* & \alpha_2^* \\ \alpha^{***} & \alpha_3^* & 0 & 0 \\ -\alpha_2^* + 2\alpha_3^* & 0 & 0 & 0 \\ \alpha_1^* - \alpha_2^* + \alpha_3^* & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $Aut(A_{01}^4)$ on the subspace $\langle \sum_{i=1}^3 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^3 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = x^4 \alpha_1, \quad \alpha_2^* = x^4 \alpha_2, \quad \alpha_3^* = x^4 \alpha_3.$$

For 1-dimensional central extensions we have the following new cases:

1. If $\alpha_1 \neq 0, \alpha_2 = 0, \alpha_3 = 0$, then $x = \frac{1}{\sqrt[4]{\alpha_1}}$, we have the representative $\langle \nabla_1 \rangle$;
2. If $\alpha_2 \neq 0, \alpha_3 = 0$, then $x = \frac{1}{\sqrt[4]{\alpha_2}}, \alpha = \frac{\alpha_1}{\alpha_2}$ we have the representative $\langle \alpha \nabla_1 + \nabla_2 \rangle$;
3. If $\alpha_3 \neq 0$, then $x = \frac{1}{\sqrt[4]{\alpha_3}}, \alpha = \frac{\alpha_1}{\alpha_3}, \beta = \frac{\alpha_2}{\alpha_3}$ we have the representative $\langle \alpha \nabla_1 + \beta \nabla_2 + \nabla_3 \rangle$.

From here, we have new 5-dimensional one generated assosymmetric algebras constructed from A_{01}^4:

A_{03}^5: $e_1e_1 = e_2 \quad e_1e_2 = e_4 \quad e_2e_1 = e_3 \quad e_1e_3 = e_5 \quad e_4e_1 = e_5$;

$A_{04}^5(\alpha)$: $e_1e_1 = e_2 \quad e_1e_2 = e_4 \quad e_2e_1 = e_3 \quad e_1e_3 = \alpha e_5 \quad e_1e_4 = e_5 \quad e_3e_1 = -e_5 \quad e_4e_1 = (\alpha - 1)e_5$;

$A_{05}^5(\alpha, \beta)$: $e_1e_1 = e_2 \quad e_1e_2 = e_4 \quad e_2e_1 = e_3 \quad e_1e_3 = \alpha e_5 \quad e_1e_4 = \beta e_5 \quad e_2e_2 = e_5 \quad e_3e_1 = (2 - \beta)e_5 \quad e_4e_1 = (\alpha - \beta + 1)e_5$.

For 2-dimensional central extensions we consider the vector space generated by the following two cocycles

$$\theta_1 = \alpha_1 \nabla_1 + \alpha_2 \nabla_2 + \alpha_3 \nabla_3,$$

$$\theta_2 = \beta_1\nabla_1 + \beta_2\nabla_2.$$

Here we have the following cases:

1. If $\alpha_3 = 0$, then we have the representative $\langle\nabla_1, \nabla_2\rangle$;
2. If $\alpha_3 \neq 0, \beta_1 \neq 0, \beta_2 = 0$, then we have the representative $\langle\nabla_1, \alpha\nabla_2 + \nabla_3\rangle$;
3. If $\alpha_3 \neq 0, \beta_2 \neq 0$, then we have the representative $\langle\alpha\nabla_1 + \nabla_2, \beta\nabla_1 + \nabla_3\rangle$.

We have the following new 6-dimensional one-generated nilpotent assosymmetric algebras constructed from A_{01}^4: $A_{01}^6, A_{02}^6(\alpha),\ A_{03}^6(\alpha,\beta)$ (see Table A.5 in Appendix A).

2) Central extensions of $A_{04}^4(1)$. Let us use the following notations:

$$\nabla_1 = [\Delta_{21}], \nabla_2 = [\Delta_{14}] + [\Delta_{23}] + [\Delta_{32}] + [\Delta_{41}].$$

The automorphism group of $A_{04}^4(1)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ y & x^2 & 0 & 0 \\ z & 2xy & x^3 & 0 \\ t & 2xz + y^2 & 3yx^2 & x^4 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 0 & \alpha_2 \\ \alpha_1 & 0 & \alpha_2 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^{***} & \alpha^* & \alpha^{**} & \alpha_2^* \\ \alpha^* + \alpha_1^* & \alpha^{**} & \alpha_2^* & 0 \\ \alpha^{**} & \alpha_2^* & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $Aut\left(A_{04}^4(1)\right)$ on the subspace $\left\langle\sum_{i=1}^2 \alpha_i\nabla_i\right\rangle$ is given by $\left\langle\sum_{i=1}^2 \alpha_i^*\nabla_i\right\rangle$, where

$$\alpha_1^* = x^3\alpha_1, \quad \alpha_2^* = x^5\alpha_2.$$

For 1-dimensional central extensions note that if $\alpha_2 = 0$ then we obtain algebras with 2-dimensional annihilator. Therefore, we have two representatives $\langle\nabla_2\rangle$ and $\langle\nabla_1 + \nabla_2\rangle$ depending on whether $\alpha_1 = 0$ or not.

We have the following new 5-dimensional nilpotent assosymmetric algebras constructed from $A_{04}^4(1)$: A_{06}^5 and A_{07}^5 (see Table A.1 in Appendix A).

For 2-dimensional central extensions we have only one new 6-dimensional nilpotent assosymmetric algebras constructed from $A_{04}^4(1)$:

$$A_{04}^6: \quad e_1e_1 = e_2 \quad e_1e_2 = e_3 \quad e_1e_3 = e_4 \quad e_1e_4 = e_5 \quad e_2e_1 = e_3 + e_6$$
$$e_2e_2 = e_4 \quad e_2e_3 = e_5 \quad e_3e_1 = e_4 \quad e_3e_2 = e_5 \quad e_4e_1 = e_5.$$

Summarizing results of the previous sections, we have the first part of Theorem 2.7.

All multiplication tables of 5-dimensional one-generated nilpotent assosymmetric algebras is given in Table A.3 (see, Appendix A). All necessary information about coboundaries, cocycles and second cohomology spaces of 5-dimensional one-generated nilpotent assosymmetric algebras were calculated by the code in [41] and given in Table A.4 (see, Appendix A).

Remark 3.2.11 *Extensions of the algebras* A_{01}^5, $A_{02}^5(\alpha)_{\alpha\neq1}$, A_{03}^5, $A_{04}^5(\alpha)$ *and* $A_{05}^5(\alpha,\beta)_{\alpha\neq\frac{1}{2}\left(\beta\pm\sqrt{-2+6\beta-3\beta^2}\right)}$ *give algebras with* 2*-dimensional annihilator. Then, in the following subsections we study the central extensions of the other algebras.*

3) Central extensions of $A_{02}^5(1)$. Let us use the following notations:

$$\nabla_1 = [\Delta_{13}] - [\Delta_{41}], \nabla_2 = [\Delta_{14}] + [\Delta_{31}] - [\Delta_{41}],$$

$$\nabla_3 = [\Delta_{15}] - [\Delta_{23}] - [\Delta_{32}] + [\Delta_{51}].$$

The automorphism group of A_{07}^5 consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ y & x^2 & 0 & 0 & 0 \\ z & 2xy & x^3 & 0 & 0 \\ t & xy & 0 & x^3 & 0 \\ w & -y^2-2xz & -3x^2y & 0 & x^4 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 0 & -\alpha_3 & 0 & 0 \\ \alpha_2 & -\alpha_3 & 0 & 0 & 0 \\ -\alpha_1-\alpha_2 & 0 & 0 & 0 & 0 \\ \alpha_3 & 0 & 0 & 0 & 0 \end{pmatrix} \phi =$$

$$\begin{pmatrix} \alpha^{****} & \alpha^{***} & \alpha_1^*+\alpha^* & \alpha_2^* & \alpha_3^* \\ \alpha^{**} & \alpha^* & -\alpha_3^* & 0 & 0 \\ \alpha_2^*+\alpha^* & -\alpha_3^* & 0 & 0 & 0 \\ -\alpha_1^*-\alpha_2^* & 0 & 0 & 0 & 0 \\ \alpha_3^* & 0 & 0 & 0 & 0 \end{pmatrix}$$

we have that the action of $Aut(A^5_{07})$ on $\langle \sum_{i=1}^{3} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{3} \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = x^4\alpha_1, \quad \alpha_2^* = x^4\alpha_2, \quad \alpha_3^* = x^5\alpha_3.$$

We have the following case:

1. If $\alpha_2 \neq 0$, then choosing $x = \frac{\alpha_2}{\alpha_3}$ we have the representative $\langle \alpha\nabla_1 + \nabla_2 + \nabla_3 \rangle$;
2. If $\alpha_2 = 0$, we have two representatives $\langle \nabla_3 \rangle$ and $\langle \nabla_1 + \nabla_3 \rangle$ depending on whether $\alpha_1 = 0$ or not.

Consequently, we have the following algebras from $A^5_{02}(1)$:

$A^6_{05}(\alpha)$:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = -e_5 + \alpha e_6$
	$e_1e_4 = e_6$	$e_1e_5 = e_6$	$e_2e_1 = e_3 + e_4$
	$e_2e_2 = -e_5$	$e_2e_3 = -e_6$	$e_3e_1 = -e_5 + e_6$
	$e_3e_2 = -e_6$	$e_4e_1 = -(\alpha+1)e_6$	$e_5e_1 = e_6$;
A^6_{06}:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = -e_5 + e_6$
	$e_1e_5 = e_6$	$e_2e_1 = e_3 + e_4$	$e_2e_2 = -e_5$
	$e_2e_3 = -e_6$	$e_3e_1 = -e_5$	$e_3e_2 = -e_6$
	$e_4e_1 = -e_6$	$e_5e_1 = e_6$;	
A^6_{07}:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = -e_5$
	$e_1e_5 = e_6$	$e_2e_1 = e_3 + e_4$	$e_2e_2 = -e_5$
	$e_2e_3 = -e_6$	$e_3e_1 = -e_5$	$e_3e_2 = -e_6$
	$e_5e_1 = e_6$.		

4) **Central extensions of $\boldsymbol{A^5_{05}(\alpha,\beta)}$.** Here we will consider the special cases for $\alpha = \frac{1}{2}\left(\beta \pm \sqrt{-2+6\beta-3\beta^2}\right)$.

The automorphism group of $A^5_{05}(\alpha,\beta)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ \frac{y}{x} & x^2 & 0 & 0 & 0 \\ z & y & x^3 & 0 & 0 \\ t & y & 0 & x^3 & 0 \\ v & \frac{x^3\left((2-\beta+\alpha)z+(1+\alpha)t\right)+y^2}{x^2} & (\alpha-2\beta+4)xy & (\alpha+\beta+1)xy & x^4 \end{pmatrix}.$$

Let use the following notations:

$$\nabla_1 = [\Delta_{14}] - [\Delta_{31}] - [\Delta_{41}], \nabla_2 = [\Delta_{13}] + [\Delta_{41}],$$

$$\nabla_3 = (2\beta - 1)[\Delta_{15}] + (2\alpha\beta - 2\beta + 1)[\Delta_{23}] + (\alpha + 2\beta^2 - 3\beta + 1)[\Delta_{24}] +$$

$$(3\alpha - 2\alpha\beta + 2\beta^2 - 3\beta + 1)[\Delta_{32}] + (2\alpha - 2\alpha\beta + 2\beta - 1)[\Delta_{42}]$$

$$+(2\alpha - 2\beta + 1)[\Delta_{51}].$$

So,

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_2 & \alpha_1 & (2\beta-1)\alpha_3 \\ 0 & 0 & (2\alpha\beta - 2\beta + 1)\alpha_3 & (\alpha + 2\beta^2 - 3\beta + 1)\alpha_3 & 0 \\ -\alpha_1 & (3\alpha - 2\alpha\beta + 2\beta^2 - 3\beta + 1)\alpha_3 & 0 & 0 & 0 \\ -\alpha_1 + \alpha_2 & (2\alpha - 2\alpha\beta + 2\beta - 1)\alpha_3 & 0 & 0 & 0 \\ (2\alpha - 2\beta + 1)\alpha_3 & 0 & 0 & 0 & 0 \end{pmatrix} \phi =$$

$$=$$

$$\begin{pmatrix} \alpha^{****} & \alpha^{***} & \alpha\alpha^* + \alpha_2^* & \beta\alpha^* + \alpha_1^* & (2\beta-1)\alpha_3^* \\ \alpha^{**} & \alpha^* & (2\alpha\beta - 2\beta + 1)\alpha_3^* & (\alpha + 2\beta^2 - 3\beta + 1)\alpha_3^* & 0 \\ (2-\beta)\alpha^* - \alpha_1^* & (3\alpha - 2\alpha\beta + 2\beta^2 - 3\beta + 1)\alpha_3^* & 0 & 0 & 0 \\ (1 - \beta + \alpha)\alpha^* - \alpha_1^* + +\alpha_2^* & (2\alpha - 2\alpha\beta + 2\beta - 1)\alpha_3^* & 0 & 0 & 0 \\ (2\alpha - 2\beta + 1)\alpha_3^* & 0 & 0 & 0 & 0 \end{pmatrix}$$

we have that the action of $Aut\left(A_{05}^5(\alpha,\beta)\right)$ on the subspace $\langle \sum_{i=1}^3 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^3 \alpha_i^* \nabla_i \rangle$, where

$$\begin{aligned} \alpha_1^* &= x^4\alpha_1 - \beta(\beta-2)(4\beta - 2\alpha - 2)\alpha_3 x^2 y, \\ \alpha_2^* &= x^4\alpha_2 - \big(\beta(\beta-2)(2\beta-1) + \alpha(2\beta^2 - 4\beta + 3)\big)\alpha_3 x^2 y, \\ \alpha_3^* &= x^5\alpha_3. \end{aligned}$$

We are interested only in the cases with $\alpha_3 \neq 0$. Now we obtain the following cases:

1. For $\beta(\beta-2)(2\beta-1) + \alpha(2\beta^2 - 4\beta + 3) \neq 0$:

(a) If $2\beta(\beta-2)(2\beta-\alpha-1)\alpha_2 = \alpha_1\big(\beta(\beta-2)(2\beta-1) + \alpha(2\beta^2 - 4\beta + 3)\big)$, then by choosing $x = \frac{1}{\sqrt[5]{\alpha_3}}$ and $y = \frac{\alpha_2 x^2}{\beta(\beta-2)(2\beta-1)+\alpha(2\beta^2-4\beta+3)}$, we have the representative $\langle \nabla_3 \rangle$;

(b) If $2\beta(\beta-2)(2\beta-\alpha-1)\alpha_2 \neq \alpha_1\big(\beta(\beta-2)(2\beta-1) + \alpha(2\beta^2 - 4\beta + 3)\big)$, then by choosing

$$x = \frac{\alpha_1\big(\alpha(2\beta^2-4\beta+3)+\beta(\beta-2)(2\beta-1)\big)+2\alpha_2\beta(\beta-2)(\alpha-2\beta+1)}{\beta(\beta-2)(2\beta-1)+\alpha(2\beta^2-4\beta+3)}$$

and

$$y = \frac{\alpha_2 x^2}{\beta(\beta-2)(2\beta-1)+\alpha(2\beta^2-4\beta+3)},$$

we obtain the representative $\langle \nabla_1 + \nabla_3 \rangle$.

From the above cases we have new parametric algebras $A_8^6(\beta)$, $A_9^6(\beta)$, $A_{10}^6(\beta)$, $A_{11}^6(\beta)$ (see Table A.5 in Appendix A).

2. The condition $\beta = 1$, for $\alpha = \frac{1}{2}\left(\beta + \sqrt{-2+6\beta-3\beta^2}\right)$ gives $\alpha = 1$, that is $A_{05}^5(1,1)$. The base of the second cohomology of this algebra spanned by elements:

$$\nabla_1 = [\Delta_{14}] - [\Delta_{31}] - [\Delta_{41}], \nabla_2 = [\Delta_{13}] + [\Delta_{41}],$$

$$\nabla_3 = [\Delta_{15}] + [\Delta_{23}] + [\Delta_{24}] + [\Delta_{32}] + [\Delta_{42}] + [\Delta_{51}].$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_2 & \alpha_1 & \alpha_3 \\ 0 & 0 & \alpha_3 & \alpha_3 & 0 \\ -\alpha_1 & \alpha_3 & 0 & 0 & 0 \\ \alpha_2 - \alpha_1 & \alpha_3 & 0 & 0 & 0 \\ \alpha_3 & 0 & 0 & 0 & 0 \end{pmatrix} \phi =$$

$$\begin{pmatrix} \alpha^{****} & \alpha^{**} & \alpha^* + \alpha_2^* & \alpha^* + \alpha_1^* & \alpha_3^* \\ \alpha^{***} & \alpha^* & \alpha_3^* & \alpha_3^* & 0 \\ \alpha^* - \alpha_1^* & \alpha_3^* & 0 & 0 & 0 \\ \alpha^* - \alpha_1^* + \alpha_2^* & \alpha_3^* & 0 & 0 & 0 \\ \alpha_3^* & 0 & 0 & 0 & 0 \end{pmatrix}$$

we have that the action of $Aut\left(A_{05}^5(1,1)\right)$ on the subspace $\left\langle \sum_{i=1}^3 \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum_{i=1}^3 \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = x^4 \alpha_1, \quad \alpha_2^* = x^4 \alpha_2, \quad \alpha_3^* = \alpha_3 x^5.$$

We are interested only in $\alpha_3 \neq 0$, then we have the following cases:

(a) If $\alpha_2 \neq 0$, then for $x = \frac{\alpha_2}{\alpha_3}$, $\alpha = \frac{\alpha_1}{\alpha_2}$ we have the representative $\langle \alpha\nabla_1 + \nabla_2 + \nabla_3 \rangle$.

(b) If $\alpha_2 = 0$, then also we have two cases:

i. If $\alpha_1 \neq 0$, then $x = \frac{\alpha_1}{\alpha_3}$, and we have the representative $\langle \nabla_1 + \nabla_3 \rangle$;

ii. If $\alpha_1 = 0$, then $x = \frac{1}{\sqrt[5]{\alpha_3}}$, and we have the representative $\langle \nabla_3 \rangle$;

Consequently, we have the following algebras from $A_{05}^5(1,1)$: $A_{08}^6(1)$, $A_{09}^6(1)$, $A_{12}^6(\alpha)$ (see Table A.1 in Appendix A).

3. The condition $\beta = \frac{3}{2}$ gives $\alpha = 1$ for $\alpha = \frac{1}{2}\left(\beta + \sqrt{-2+6\beta-3\beta^2}\right)$, that is $A_{05}^5\left(1,\frac{3}{2}\right)$. So, the second cohomology space of $A_{05}^5\left(1,\frac{3}{2}\right)$ spanned by elements:

$$\nabla_1 = [\Delta_{14}] - [\Delta_{31}] - [\Delta_{41}], \nabla_2 = [\Delta_{13}] + [\Delta_{41}],$$

$$\nabla_3 = 2[\Delta_{15}] + [\Delta_{23}] + 2[\Delta_{24}] + [\Delta_{32}] + [\Delta_{42}].$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_2 & \alpha_1 & 2\alpha_3 \\ 0 & 0 & \alpha_3 & 2\alpha_3 & 0 \\ -\alpha_1 & \alpha_3 & 0 & 0 & 0 \\ \alpha_2 - \alpha_1 & \alpha_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \phi =$$

$$\begin{pmatrix} \alpha^{****} & \alpha^{***} & \alpha_2^* + \alpha^* & \alpha_1^* + 3\alpha^* & 2\alpha_3^* \\ \alpha^{**} & 2\alpha^* & \alpha_3^* & 2\alpha_3^* & 0 \\ \alpha^* - \alpha_1^* & \alpha_3^* & 0 & 0 & 0 \\ \alpha_2^* - \alpha_1^* + \alpha^* & \alpha_3^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

we have that the action of $Aut\left(A_{05}^5\left(1,\frac{3}{2}\right)\right)$ on the subspace $\left\langle \sum_{i=1}^3 \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum_{i=1}^3 \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = x^4\alpha_1 + \frac{3}{2}x^3 y \alpha_3, \quad \alpha_2^* = x^4\alpha_2, \quad \alpha_3^* = x^5\alpha_3.$$

Since $\alpha_3 \neq 0$, and choosing $y = -\frac{2x^2\alpha_1}{3\alpha_3}$, we have the representatives $\langle \nabla_3 \rangle$ and $\langle \nabla_2 + \nabla_3 \rangle$, depending on whether $\alpha_2 = 0$ or not.

We have the following new 6-dimensional algebras constructed from $A_{05}^5\left(1,\frac{3}{2}\right)$: $A_{08}^6\left(\frac{3}{2}\right)$, A_{13}^6 (see Table A.5 in Appendix A).

5) Central extensions of $A_{05}^5\left(0,\frac{1}{2}\right)$. If $\beta = \frac{1}{2}$ for $\alpha = \frac{1}{2}\left(\beta - \sqrt{-2+6\beta-3\beta^2}\right)$ gives $\alpha = 0$, that is $A_{05}^5\left(0,\frac{1}{2}\right)$. So, the second cohomology space of $A_{05}^5\left(0,\frac{1}{2}\right)$ spanned by elements:

$$\nabla_1 = [\Delta_{14}] - [\Delta_{31}] - [\Delta_{41}],$$

$$\nabla_2 = [\Delta_{13}] + [\Delta_{41}],$$

$$\nabla_3 = 2[\Delta_{15}] - 3[\Delta_{23}] - 2[\Delta_{24}] - 3[\Delta_{32}] + [\Delta_{42}] - 4[\Delta_{51}].$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_2 & \alpha_1 & 2\alpha_3 \\ 0 & 0 & -3\alpha_3 & -2\alpha_3 & 0 \\ -\alpha_1 & -3\alpha_3 & 0 & 0 & 0 \\ \alpha_2 - \alpha_1 & \alpha_3 & 0 & 0 & 0 \\ -4\alpha_3 & 0 & 0 & 0 & 0 \end{pmatrix} \phi =$$

$$= \begin{pmatrix} \alpha^{****} & \alpha^{**} & \alpha_2^* & \alpha_1^* + \alpha^* & 2\alpha_3^* \\ \alpha^{***} & 2\alpha^* & -3\alpha_3^* & -2\alpha_3^* & 0 \\ -\alpha_1^* + 3\alpha^* & -3\alpha_3^* & 0 & 0 & 0 \\ \alpha_2^* - \alpha_1^* + \alpha^* & \alpha_3^* & 0 & 0 & 0 \\ -4\alpha_3^* & 0 & 0 & 0 & 0 \end{pmatrix}$$

we have that the action of $Aut\left(A_{05}^5\left(0,\frac{1}{2}\right)\right)$ on the subspace $\langle\sum_{i=1}^3 \alpha_i \nabla_i\rangle$ is given by $\langle\sum_{i=1}^3 \alpha_i^* \nabla_i\rangle$, where

$$\alpha_1^* = x^4\alpha_2 + \frac{9}{2}x^3y\alpha_3, \quad \alpha_2^* = x^4\alpha_1 + 3x^3y\alpha_3, \quad \alpha_3^* = x^5\alpha_3.$$

We are interested in $\alpha_3 \neq 0$, then we have the following cases:

1. If $3\alpha_1 - 2\alpha_2 = 0$, then $x = \frac{1}{\sqrt[5]{\alpha_3}}$ and $y = -\frac{x\alpha_1}{3\alpha_3}$, we have the representative $\langle\nabla_3\rangle$;
2. If $3\alpha_1 - 2\alpha_2 \neq 0$, then $x = \frac{-3\alpha_1 + 2\alpha_2}{2\alpha_3}, y = -\frac{x\alpha_1}{3\alpha_3}$ and we have the representative $\langle\nabla_2 + \nabla_3\rangle$.

We have the following new 6-dimensional algebras constructed from $A_{05}^5\left(0,\frac{1}{2}\right)$: A_{14}^6 and A_{15}^6 (see Table A.5 in Appendix A).

6) Central extensions of $\mathbf{A_{06}^5}$. Let us use the following notations:

$$\nabla_1 = [\Delta_{21}], \nabla_2 = [\Delta_{15}] + [\Delta_{24}] + [\Delta_{33}] + [\Delta_{42}] + [\Delta_{51}].$$

The automorphism group of A_{06}^5 consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ y & x^2 & 0 & 0 & 0 \\ z & 2xy & x^3 & 0 & 0 \\ v & 2xz + y^2 & 3x^2y & x^4 & 0 \\ w & 2xv + 2yz & 3x^2z + 3xy^2 & 4x^3y & x^5 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 0 & 0 & \alpha_2 \\ \alpha_1 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^{****} & \alpha^* & \alpha^{**} & \alpha^{***} & \alpha_2^* \\ \alpha_1^* + \alpha^* & \alpha^{**} & \alpha^{***} & \alpha_2^* & 0 \\ \alpha^{**} & \alpha^{***} & \alpha_2^* & 0 & 0 \\ \alpha^{***} & \alpha_2^* & 0 & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $Aut(A_{06}^5)$ on the subspace $\langle \sum_{i=1}^2 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^2 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = x^3\alpha_1, \quad \alpha_2^* = x^6\alpha_2.$$

We suppose that $\alpha_2 \neq 0$, otherwise obtained algebra gives an algebra with 2-dimensional annihilator. Therefore, consider the following cases:

1. If $\alpha_1 = 0$, then $x = \frac{1}{\sqrt[6]{\alpha_2}}$, we have the representative $\langle \nabla_2 \rangle$;
2. If $\alpha_1 \neq 0$, then $x = \sqrt[3]{\frac{\alpha_1}{\alpha_2}}$, we have the representative $\langle \nabla_1 + \nabla_2 \rangle$.

Hence, we have the following new algebras: A_{16}^6, A_{17}^6 (see Table A.5 in Appendix A).

7) Central extensions of $\mathbf{A_{07}^5}$. Let us use the following notations:

$$\nabla_1 = [\Delta_{21}], \nabla_2 = [\Delta_{15}] + 2[\Delta_{22}] + [\Delta_{24}] + 3[\Delta_{31}] + [\Delta_{33}] + [\Delta_{42}] + [\Delta_{51}].$$

The automorphism group of A_{07}^5 consists of invertible matrices of the form

$$\phi_i = \begin{pmatrix} (-1)^k & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ y & (-1)^k 2x & (-1)^k & 0 & 0 \\ z & x^2 + (-1)^k 2y & 3x & 1 & 0 \\ t & 2xy + (-1)^k(x + 2z) & (-1)^k 3x^2 + 3y & (-1)^k 4x & (-1)^k \end{pmatrix},$$

where $k \in 1,2$. Since

$$\phi_i^T\begin{pmatrix} 0 & 0 & 0 & 0 & \alpha_2 \\ \alpha_1 & 2\alpha_2 & 0 & \alpha_2 & 0 \\ 3\alpha_2 & 0 & \alpha_2 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 & 0 \end{pmatrix}\phi_i = \begin{pmatrix} \alpha^{****} & \alpha^{*} & \alpha^{**} & \alpha^{***} & \alpha_2^* \\ \alpha_1^* + \alpha^* & 2\alpha_2^* + \alpha^{**} & \alpha^{***} & \alpha_2^* & 0 \\ 3\alpha_2^* & \alpha^{***} & \alpha_2^* & 0 & 0 \\ \alpha^{***} & \alpha_2^* & 0 & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $Aut(A_{07}^5)$ on the subspace $\left\langle \sum_{i=1}^{2} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum_{i=1}^{2} \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = (-1)^i \alpha_1 - 6x\alpha_2, \quad \alpha_2^* = \alpha_2.$$

We have only one non-trivial orbit with the representative $\langle \nabla_2 \rangle$, and get the algebra A_{18}^6 (see Table A.5 in Appendix A).

Summarizing results we have the second part of Theorem 3.2.7.

CONCLUSION

In conclusion, the dissertation work has focused on two classical problems in the study of nonassociative algebras, specifically, the study of nonassociative algebras under a commutator and the classification of finitely dimensional nonassociative algebras.

Firstly, a criterion was found for determining the Lie elements in a free Zinbiel algebra. This result is of particular importance as it allows for the identification of Lie elements in a free Zinbiel algebra, which is a fundamental step in understanding the structure and properties of these algebras.

Secondly, a basis for special Tortkara algebras was constructed. This result provides a foundation for further study of these algebras and can be used to develop new techniques and methods. Additionally, it was shown that there exists an exceptional homomorphic image of a free special Tortkara algebra with three generators, and it was proved that any homomorphic image of a free special Tortkara algebra with two generators is special.

Thirdly, it has been proved that there is no special identity with two generators. This result has implications for the study of special identities in special Tortkara algebras and can be used to future research in this area.

Fourthly, an algebraic classification of nilpotent 4-dimensional assosymmetric algebras was constructed. This result provides a comprehensive understanding of the structure and properties of these algebras and can be used to guide further research in this area. Additionally, an algebraic classification of nilpotent 5- and 6-dimensional assosymmetric algebras with one generator was constructed.

Finally, algorithms were provided with code written in Wolfram Mathematica to simplify the computational aspects of the classification problem of nilpotent algebras. The use of Wolfram Mathematica allowed for efficient and accurate computations, and the authors partially used the "solve" function, a symbolic solver built into Wolfram Mathematica, when working with a system of polynomial equations. This is the main function that takes up most of the compilation time. The codes written by the authors in other software, including Matlab and Python, gave the worst results in terms of running time and, in some cases, failed to provide any solutions.

In summary, the results obtained in this dissertation have advanced our understanding of nonassociative algebras and have provided new techniques and methods. These results can be applied to further study of Zinbiel algebras under commutator and can be used in special courses on the theory of free and finite-dimensional algebras. The work also highlights the importance of using appropriate software tools when working with symbolic nonlinear equations and the potential for improving the performance of these tools. Overall, this dissertation work has made a contribution to the field of nonassociative algebras and sets the stage for future research in this area.

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APPENDIX A

Table A.1 – The list of 4-dimensional nilpotent "pure" assosymmetric algebras.

A_{01}^4	$e_1e_1 = e_2$	$e_1e_2 = e_3$	
$A_{02}^4(\alpha)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_2e_1 = \alpha e_3$
A_{03}^4	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_3$
$A_{04}^4(\alpha)$	$e_1e_1 = e_2$ $e_3e_3 = e_4$	$e_1e_2 = e_4$	$e_2e_1 = \alpha e_4$
A_{05}^4	$e_1e_1 = e_2$ $e_1e_3 = e_4$	$e_1e_2 = e_4$ $e_3e_3 = e_4$	$e_2e_1 = \alpha e_4$
$A_{06}^4(\alpha \neq 0)$	$e_1e_1 = e_2$ $e_2e_1 = \alpha e_4$	$e_1e_2 = e_4$	$e_1e_3 = e_4$
A_{07}^4	$e_1e_1 = e_2$	$e_2e_1 = e_4$	$e_3e_3 = e_4$
A_{08}^4	$e_1e_1 = e_2$	$e_2e_1 = e_4$	$e_1e_3 = e_4$
A_{09}^4	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_3e_1 = e_4$
A_{10}^4	$e_1e_1 = e_3$ $e_2e_2 = e_3$	$e_2e_1 = e_4$ $e_2e_3 = e_4$	$e_1e_3 = e_4$ $e_3e_2 = -e_4$
A_{11}^4	$e_1e_1 = e_3$ $e_2e_2 = e_3$	$e_2e_1 = e_3 + e_4$	$e_1e_3 = e_4$
A_{12}^4	$e_1e_1 = e_3$ $e_2e_2 = e_3 + e_4$ $e_3e_2 = -e_4$	$e_2e_1 = e_3$ $e_2e_3 = \frac{1+\sqrt{3}i}{2}e_4$	$e_1e_3 = e_4$ $e_3e_1 = \frac{-1+\sqrt{3}i}{2}e_4$
A_{13}^4	$e_1e_1 = e_3$ $e_2e_2 = e_3$ $e_3e_2 = -e_4$	$e_2e_1 = e_3$ $e_2e_3 = \frac{1+\sqrt{3}i}{2}e_4$	$e_1e_3 = e_4$ $e_3e_1 = \frac{-1+\sqrt{3}i}{2}e_4$

Table A.1 – continued from previous page

A_{14}^4	$e_1e_1 = e_3$ $\quad$ $e_2e_1 = e_3$ $\quad$ $e_1e_3 = e_4$ $e_2e_2 = e_3 + e_4$ $\quad$ $e_2e_3 = \frac{1-\sqrt{3}i}{2}e_4$ $\quad$ $e_3e_1 = \frac{-1-\sqrt{3}i}{2}e_4$ $e_3e_2 = -e_4$
A_{15}^4	$e_1e_1 = e_3$ $\quad$ $e_2e_1 = e_3$ $\quad$ $e_1e_3 = e_4$ $e_2e_2 = e_3$ $\quad$ $e_2e_3 = \frac{1-\sqrt{3}i}{2}e_4$ $\quad$ $e_3e_1 = \frac{-1-\sqrt{3}i}{2}e_4$ $e_3e_2 = -e_4$
A_{16}^4	$e_1e_1 = e_2$ $\quad$ $e_1e_2 = e_4$ $\quad$ $e_1e_3 = e_4$ $e_2e_1 = e_3$ $\quad$ $e_2e_2 = -e_4$ $\quad$ $e_3e_1 = -2e_4$
A_{17}^4	$e_1e_1 = e_2$ $\quad$ $e_1e_3 = e_4$ $\quad$ $e_2e_1 = e_3$ $e_2e_2 = -e_4$ $\quad$ $e_3e_1 = -2e_4$
$A_{18}^4(\alpha)$	$e_1e_1 = e_2$ $\quad$ $e_1e_2 = e_3$ $\quad$ $e_1e_3 = (2-\alpha)e_4$ $e_2e_1 = \alpha e_3$ $\quad$ $e_2e_2 = (\alpha^2-\alpha+1)e_4$ $\quad$ $e_3e_1 = (2\alpha-1)e_4$
A_{19}^4	$e_1e_1 = e_2$ $\quad$ $e_1e_2 = e_3$ $\quad$ $e_1e_3 = -2e_4$ $e_2e_1 = e_4$ $\quad$ $e_2e_2 = -e_4$ $\quad$ $e_3e_1 = -2e_4$
A_{20}^4	$e_1e_1 = e_2$ $\quad$ $e_1e_2 = e_3$ $\quad$ $e_1e_3 = -3e_4$ $e_2e_1 = -e_3 + e_4$ $\quad$ $e_2e_2 = -3e_4$ $\quad$ $e_3e_1 = 3e_4$

Cohomology spaces of 4-dimensional one-generated assosymmetric algebras. All multiplication tables of four-dimensional one-generated nilpotent assosymmetric algebras is given in Table 2. In the present table we collect all usefull information about $\mathrm{Z}^2, \mathrm{B}^2$ and H^2 spaces for all four-dimensional one-generated nilpotent assosymmetric algebras that were counted via code in [41].

Table A.2 – Cohomology spaces of 4-dimensional one-generated nilpotent assosymmetric algebras

$Z^2(A_{01}^4)$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} + \Delta_{41}, \Delta_{14} - \Delta_{31} - \Delta_{41}, \Delta_{22} + 2\Delta_{31} + \Delta_{41} \rangle$
$B^2(A_{01}^4)$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21} \rangle$
$H^2(A_{01}^4)$	$\langle [\Delta_{13}] + [\Delta_{41}], [\Delta_{14}] - [\Delta_{31}] - [\Delta_{41}], [\Delta_{22}] + 2[\Delta_{31}] + [\Delta_{41}] \rangle$
$Z^2(A_{02}^4)$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} - \Delta_{22} - 2\Delta_{31} \rangle$
$B^2(A_{02}^4)$	$\langle \Delta_{11}, \Delta_{21}, \Delta_{12} + \Delta_{13} - \Delta_{22} - 2\Delta_{31} \rangle$
$H^2(A_{02}^4)$	$\langle [\Delta_{12}] \rangle$
$Z^2(A_{03}^4)$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} - \Delta_{22} - 2\Delta_{31} \rangle$
$B^2(A_{03}^4)$	$\langle \Delta_{11}, \Delta_{21}, \Delta_{13} - \Delta_{22} - 2\Delta_{31} \rangle$
$H^2(A_{03}^4)$	$\langle [\Delta_{12}] \rangle$
$Z^2(A_{04}^4(\alpha)_{\alpha\neq 1})$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, (2-\alpha)\Delta_{13} + (\alpha^2 - \alpha + 1)\Delta_{22} + (2\alpha - 1)\Delta_{31} \rangle$
$B^2(A_{04}^4(\alpha)_{\alpha\neq 1})$	$\langle \Delta_{11}, \Delta_{12} + \alpha\Delta_{21}, (2-\alpha)\Delta_{13} + (\alpha^2 - \alpha + 1)\Delta_{22} + (2\alpha - 1)\Delta_{31} \rangle$
$H^2(A_{04}^4(\alpha)_{\alpha\neq 1})$	$\langle [\Delta_{12}] \rangle$
$Z^2(A_{04}^4(1))$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} + \Delta_{22} + \Delta_{31}, \Delta_{14} + \Delta_{23} + \Delta_{32} + \Delta_{41} \rangle$
$B^2(A_{04}^4(1))$	$\langle \Delta_{11}, \Delta_{12} + \Delta_{21}, \Delta_{13} + \Delta_{22} + \Delta_{31} \rangle$
$H^2(A_{04}^4(1))$	$\langle [\Delta_{21}], [\Delta_{14}] + [\Delta_{23}] + [\Delta_{32}] + [\Delta_{41}] \rangle$
$Z^2(A_{05}^4)$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, 2\Delta_{13} + \Delta_{22} - \Delta_{31} \rangle$
$B^2(A_{05}^4)$	$\langle \Delta_{11}, \Delta_{12}, -2\Delta_{13} + \Delta_{21} - \Delta_{22} + \Delta_{31} \rangle$
$H^2(A_{05}^4)$	$\langle [\Delta_{21}] \rangle$
$Z^2(A_{06}^4)$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} + \Delta_{22} - \Delta_{31} \rangle$
$B^2(A_{06}^4)$	$\langle \Delta_{11}, \Delta_{12} - \Delta_{21}, -3\Delta_{13} + \Delta_{21} - 3\Delta_{22} + 3\Delta_{31} \rangle$
$H^2(A_{06}^4)$	$\langle [\Delta_{13}] + [\Delta_{22}] - [\Delta_{31}] \rangle$

Table A.3 – The list of 5-dimensional nilpotent “pure” assosymmetric algebras

A_{01}^5	$e_1e_1 = e_2 \quad e_1e_2 = e_4 \quad e_1e_3 = e_5$ $e_2e_1 = e_3 \quad e_2e_2 = -e_5 \quad e_3e_1 = -2e_5$
$A_{02}^5(\alpha)$	$e_1e_1 = e_2 \quad e_1e_2 = e_3 \quad e_1e_3 = (\alpha-2)e_5 \quad e_2e_1 = \alpha\, e_3 + e_4$ $e_2e_2 = (\alpha-\alpha^2-1)e_5 \quad e_3e_1 = (1-2\alpha)e_5$
A_{03}^5	$e_1e_1 = e_2 \quad e_1e_2 = e_4 \quad e_1e_3 = e_5 \quad e_2e_1 = e_3 \quad e_4e_1 = e_5$
$A_{04}^5(\alpha)$	$e_1e_1 = e_2 \quad e_1e_2 = e_4 \quad e_1e_3 = \alpha e_5 \quad e_1e_4 = e_5$ $e_2e_1 = e_3 \quad e_3e_1 = -e_5 \quad e_4e_1 = (\alpha-1)\, e_5$
$A_{04}^5(\alpha,\beta)$	$e_1e_1 = e_2 \quad e_1e_2 = e_4 \quad e_1e_3 = \alpha e_5 \quad e_1e_4 = \beta e_5$ $e_2e_1 = e_3 \quad e_2e_2 = e_5 \quad e_3e_1 = (2-\beta)e_5 \quad e_4e_1 = (\alpha-\beta+1)e_5$
A_{03}^5	$e_1e_1 = e_2 \quad e_1e_2 = e_3 \quad e_1e_3 = e_4 \quad e_1e_4 = e_5 \quad e_2e_1 = e_3$ $e_2e_2 = e_4 \quad e_2e_3 = e_5 \quad e_3e_1 = e_4 \quad e_3e_2 = e_5 \quad e_4e_1 = e_5$
A_{03}^5	$e_1e_1 = e_2 \quad e_1e_2 = e_3 \quad e_1e_3 = e_4 \quad e_1e_4 = e_5 \quad e_2e_1 = e_3 + e_5$ $e_2e_2 = e_4 \quad e_2e_3 = e_5 \quad e_3e_1 = e_4 \quad e_3e_2 = e_5 \quad e_4e_1 = e_5$

Cohomology spaces of 5-dimensional one-generated assosymmetric algebras. All relevant information about coboundaries, cocycles and second cohomology spaces of five-dimensional one-generated nilpotent assosymmetric algebras were calculated by the code in [41] and given in the following table:

Table A.4 – Cohomology spaces of 5-dimensional one-generated nilpotent assosymmetric algebras

$\mathrm{Z}^2(A_{01}^5)$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13}+\Delta_{41}, \Delta_{22}+2\Delta_{31}+\Delta_{41}\Delta_{14}-\Delta_{31}-\Delta_{41}\rangle$
$\mathrm{B}^2(A_{01}^5)$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13}-\Delta_{22}-2\Delta_{31}\rangle$
$\mathrm{H}^2(A_{01}^5)$	$\langle [\Delta_{13}]+[\Delta_{41}], [\Delta_{14}]-[\Delta_{31}]+[\Delta_{41}]\rangle$
$\mathrm{Z}^2(A_{02}^5(\alpha \neq 1))$	$\left\langle \begin{matrix} \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13}+(1-\alpha)\Delta_{22}+(1-2\alpha)\Delta_{41}, \\ \Delta_{14}-\Delta_{22}-2\Delta_{41}, \Delta_{22}+\Delta_{31}+(2-\alpha)\Delta_{41} \end{matrix} \right\rangle$
$\mathrm{B}^2(A_{02}^5(\alpha \neq 1))$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, (\alpha-2)\Delta_{13}+(\alpha-\alpha^2-1)\Delta_{22}+(1-2\alpha)\Delta_{31}\rangle$
$\mathrm{H}^2(A_{02}^5(\alpha \neq 1))$	$\langle [\Delta_{14}]-[\Delta_{22}]-2[\Delta_{41}], [\Delta_{22}]+[\Delta_{31}]+(2-\alpha)[\Delta_{41}]\rangle$
$\mathrm{Z}^2(A_{02}^5(1))$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13}-\Delta_{41}, \Delta_{22}+\Delta_{31}+\Delta_{41}, \Delta_{14}+\Delta_{31}-\Delta_{41}, \Delta_{15}-\Delta_{23}-\Delta_{32}+\Delta_{51}\rangle$
$\mathrm{B}^2(A_{02}^5(1))$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13}+\Delta_{22}+\Delta_{31}\rangle$
$\mathrm{H}^2(A_{02}^5(1))$	$\langle [\Delta_{13}]-[\Delta_{41}], [\Delta_{14}]+[\Delta_{31}]-[\Delta_{41}], [\Delta_{15}]-[\Delta_{23}]-[\Delta_{32}]+[\Delta_{51}]\rangle$
$\mathrm{Z}^2(A_{03}^5)$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13}+\Delta_{41}, \Delta_{14}-\Delta_{31}-\Delta_{41}, \Delta_{22}+2\Delta_{31}+\Delta_{41}\rangle$

Table A.4 – continued from previous page

$\mathrm{B}^2(A^5_{03})$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} + \Delta_{41} \rangle$
$\mathrm{H}^2(A^5_{03})$	$\langle [\Delta_{14}] - [\Delta_{31}] - [\Delta_{41}], [\Delta_{22}] + 2[\Delta_{31}] + [\Delta_{41}] \rangle$
$\mathrm{Z}^2(A^5_{04}(\alpha))$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} + \Delta_{14} - \Delta_{31}, \Delta_{14} - \Delta_{31} - \Delta_{41}, \Delta_{14} + \Delta_{22} + \Delta_{31} \rangle$
$\mathrm{B}^2(A^5_{04}(\alpha))$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \alpha\Delta_{13} + \Delta_{14} - \Delta_{31} + (\alpha - 1)\Delta_{41} \rangle$
$\mathrm{H}^2(A^5_{04}(\alpha))$	$\langle [\Delta_{13}] + [\Delta_{41}], \alpha[\Delta_{13}] + 2[\Delta_{14}] + [\Delta_{22}] + (\alpha - 1)[\Delta_{41}] \rangle$
$\mathrm{Z}^2(A^5_{05}(\alpha,\beta))$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} + \Delta_{14} - \Delta_{31}, \Delta_{14} - \Delta_{31} - \Delta_{41}, \Delta_{14} + \Delta_{22} + \Delta_{31} \rangle$
$\mathrm{B}^2(A^5_{05}(\alpha,\beta))$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \alpha\Delta_{13} + \beta\Delta_{14} + \Delta_{22} + (2-\beta)\Delta_{31} + (\alpha - \beta + 1)\Delta_{41} \rangle$
$\mathrm{H}^2(A^5_{05}(\alpha,\beta))$	$\langle [\Delta_{13}] + [\Delta_{14}] - [\Delta_{31}], [\Delta_{14}] - [\Delta_{31}] - [\Delta_{41}] \rangle$
$\boldsymbol{\alpha \neq \frac{1}{2}\left(\beta \pm \sqrt{-2 + 6\beta - 3\beta^2}\right)}$	
$\mathrm{Z}^2(A^5_{05}(\alpha,\beta))$	$\left\langle \begin{array}{l} \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{14} - \Delta_{31} - \Delta_{41}, \Delta_{13} + \Delta_{41}, (2\beta - 1)\Delta_{15} + \\ +(2\beta(\alpha - 1) + 1)\Delta_{23} + (\alpha + 2\beta^2 - 3\beta + 1)\Delta_{24} + (-2\alpha\beta + 3\alpha + 2\beta^2 - 3\beta + 1)\Delta_{32} + \\ +(-2\alpha\beta + 2\alpha + 2\beta - 1)\Delta_{42} + (2\alpha - 2\beta + 1)\Delta_{51}, \Delta_{22} + 2\Delta_{31} + \Delta_{41} \end{array} \right\rangle$
$\mathrm{B}^2(A^5_{05}(\alpha,\beta))$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \alpha\Delta_{13} + \beta\Delta_{14} + \Delta_{22} + (2-\beta)\Delta_{31} + (\alpha - \beta + 1)\Delta_{41} \rangle$
$\mathrm{H}^2(A^5_{05}(\alpha,\beta))$	$\left\langle \begin{array}{l} [\Delta_{14}] - [\Delta_{31}] - [\Delta_{41}], [\Delta_{13}] + [\Delta_{41}], (2\beta - 1)[\Delta_{15}] + (2\beta(\alpha - 1) + 1)[\Delta_{23}] + (\alpha + 2\beta^2 - 3\beta + 1)[\Delta_{24}] \\ +(-2\alpha\beta + 3\alpha + 2\beta^2 - 3\beta + 1)[\Delta_{32}] + (-2\alpha\beta + 2\alpha + 2\beta - 1)[\Delta_{42}] + (2\alpha - 2\beta + 1)[\Delta_{51}] \end{array} \right\rangle$
$\alpha = \frac{1}{2}\left(\beta \pm \sqrt{-2 + 6\beta - 3\beta^2}\right)$ and $(\alpha,\beta) \neq \left(0,\frac{1}{2}\right)$	
$\mathrm{Z}^2(A^5_{05}(0,\frac{1}{2}))$	$\left\langle \begin{array}{l} \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} + \Delta_{41}, \Delta_{14} - \Delta_{31} - \Delta_{41}, \\ 2\Delta_{15} - 3\Delta_{23} - 2\Delta_{24} - 3\Delta_{32} + \Delta_{42} - 4\Delta_{51}, \Delta_{22} + 2\Delta_{31} + \Delta_{41} \end{array} \right\rangle$
$\mathrm{B}^2(A^5_{05}(0,\frac{1}{2}))$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{14} + 2\Delta_{22} + 3\Delta_{31} + \Delta_{41} \rangle$
$\mathrm{H}^2(A^5_{05}(0,\frac{1}{2}))$	$\langle [\Delta_{14}] - [\Delta_{31}] - [\Delta_{41}], [\Delta_{13}] + [\Delta_{41}], 2[\Delta_{15}] - 3[\Delta_{23}] - 2[\Delta_{24}] - 3[\Delta_{32}] + [\Delta_{42}] - 4[\Delta_{51}] \rangle$
$\mathrm{Z}^2(A^5_{06}))$	$\left\langle \begin{array}{l} \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} + \Delta_{22} + \Delta_{31}, \Delta_{14} + \Delta_{23} + \Delta_{32} + \Delta_{41}, \\ \Delta_{15} + \Delta_{24} + \Delta_{33} + \Delta_{42} + \Delta_{51} \end{array} \right\rangle$
$\mathrm{B}^2(A^5_{06}))$	$\langle \Delta_{11}, \Delta_{12} + \Delta_{21}, \Delta_{13} + \Delta_{22} + \Delta_{31}, \Delta_{14} + \Delta_{23} + \Delta_{32} + \Delta_{41} \rangle$
$\mathrm{H}^2(A^5_{06}))$	$\langle [\Delta_{21}], [\Delta_{15}] + [\Delta_{24}] + [\Delta_{33}] + [\Delta_{42}] + [\Delta_{51}] \rangle$
$\mathrm{Z}^2(A^5_{07})$	$\left\langle \begin{array}{l} \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} + \Delta_{22} + \Delta_{31}, \Delta_{14} + \Delta_{23} + \Delta_{32} + \Delta_{41}, \\ \Delta_{15} + 2\Delta_{22} + \Delta_{24} + 3\Delta_{31} + \Delta_{33} + \Delta_{42} + \Delta_{51} \end{array} \right\rangle$
$\mathrm{B}^2(A^5_{07})$	$\langle \Delta_{11}, \Delta_{12} + \Delta_{21}, \Delta_{13} + \Delta_{22} + \Delta_{31}, \Delta_{14} + \Delta_{21} + \Delta_{23} + \Delta_{32} + \Delta_{41} \rangle$
$\mathrm{H}^2(A^5_{07})$	$\langle [\Delta_{21}], [\Delta_{15}] + 2[\Delta_{22}] + [\Delta_{24}] + 3[\Delta_{31}] + [\Delta_{33}] + [\Delta_{42}] + [\Delta_{51}] \rangle$

Table A.5 – The list of 6-dimensional nilpotent “pure” assosymmetric algebras

A^6_{01}	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5$
	$e_1e_4 = e_6$	$e_2e_1 = e_3$	$e_3e_1 = -e_6$
	$e_4e_1 = e_5 - e_6$		
$A^6_{02}(\alpha)$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5$
	$e_1e_4 = \alpha e_6$	$e_2e_1 = e_3$	$e_2e_2 = e_6$
	$e_3e_1 = (2-\alpha)e_6$	$e_4e_1 = e_5 - (\alpha - 1)e_6$	

Table A.5 – continued from previous page

$A_{03}^6(\alpha,\beta)$	$e_1e_1=e_2$ $e_1e_2=e_4$ $e_1e_3=\alpha e_5+\beta e_6$ $e_1e_4=e_5$ $e_2e_1=e_3$ $e_2e_2=e_6$ $e_3e_1=-e_5+2e_6$ $e_4e_1=(\alpha-1)e_5+(\beta+1)e_6$
A_{04}^6	$e_1e_1=e_2$ $e_1e_2=e_3$ $e_1e_3=e_4$ $e_1e_4=e_6$ $e_2e_1=e_3+e_6$ $e_2e_2=e_4$ $e_2e_3=e_5$ $e_3e_1=e_4$ $e_3e_2=e_5$ $e_4e_1=e_5$
$A_{05}^6(\alpha)$	$e_1e_1=e_2$ $e_1e_2=e_3$ $e_1e_3=-e_5+\alpha e_6$ $e_1e_4=e_6$ $e_1e_5=e_6$ $e_2e_1=e_3+e_4$ $e_2e_2=-e_5$ $e_2e_3=-e_6$ $e_3e_1=-e_5+e_6$ $e_3e_2=-e_6$ $e_4e_1=-(\alpha+1)e_6$ $e_5e_1=e_6$
A_{06}^6	$e_1e_1=e_2$ $e_1e_2=e_3$ $e_1e_3=-e_5+e_6$ $e_1e_5=e_6$ $e_2e_1=e_3+e_4$ $e_2e_2=-e_5$ $e_2e_3=-e_6$ $e_3e_1=-e_5$ $e_3e_2=-e_6$ $e_5e_1=e_6$ $e_4e_1=-e_6$
A_{07}^6	$e_1e_1=e_2$ $e_1e_2=e_3$ $e_1e_3=-e_5$ $e_1e_5=e_6$ $e_2e_1=e_3+e_4$ $e_2e_2=-e_5$ $e_2e_3=-e_6$ $e_3e_1=-e_5$ $e_3e_2=-e_6$ $e_5e_1=e_6$
$A_{08}^6(\beta)$	$\alpha=\frac{1}{2}\left(\beta+\sqrt{(-2+6\beta-3\beta^2)}\right)$ $e_1e_1=e_2$ $e_1e_2=e_4$ $e_1e_3=\alpha e_5$ $e_1e_4=\beta e_5$ $e_1e_5=(2\beta-1)e_6$ $e_2e_1=e_3$ $e_2e_2=e_5$ $e_2e_3=(2\alpha\beta-2+1)e_6$ $e_2e_4=(\alpha+2\beta^2-3\beta+1)e_6$ $e_3e_1=(2-\beta)e_5$ $e_3e_2=((\alpha-\beta)(3-2\beta)+1)e_6$ $e_4e_1=(\alpha-\beta+1)e_5$ $e_4e_2=(2\alpha-2\alpha\beta+2\beta-1)e_6$ $e_5e_1=(2\alpha-2\beta+1)e_6$

Table A.5 – continued from previous page

$A^6_{09}(\beta)$	$\alpha = \frac{1}{2}\left(\beta + \sqrt{(-2+6\beta-3\beta^2)}\right), \beta \neq \frac{3}{2}$ $e_1e_1 = e_2$ $\quad e_1e_2 = e_4$ $\quad e_1e_3 = \alpha e_5$ $e_1e_4 = \beta e_5 + e_6$ $\quad e_1e_5 = (2\beta - 1)e_6$ $\quad e_2e_1 = e_3$ $e_2e_2 = e_5$ $\quad e_2e_3 = (2\alpha\beta - 2 + 1)e_6$ $e_2e_4 = (\alpha + 2\beta^2 - 3\beta + 1)e_6$ $\quad e_3e_1 = (2-\beta)e_5 - e_6$ $e_3e_2 = ((\alpha-\beta)(3-2\beta)+1)e_6$ $\quad e_4e_1 = (\alpha - \beta + 1)e_5 - e_6$ $e_4e_2 = (2\alpha - 2\alpha\beta + 2\beta - 1)e_6$ $\quad e_5e_1 = (2\alpha - 2\beta + 1)e_6$
$A^6_{10}(\beta)$	$\alpha = \frac{1}{2}\left(\beta - \sqrt{(-2+6\beta-3\beta^2)}\right), \beta \neq \frac{1}{2}$ $e_1e_1 = e_2$ $\quad e_1e_2 = e_4$ $\quad e_1e_3 = \alpha e_5$ $e_1e_4 = \beta e_5$ $\quad e_1e_5 = (2\beta - 1)e_6$ $\quad e_2e_1 = e_3$ $e_2e_2 = e_5$ $\quad e_2e_3 = (2\alpha\beta - 2 + 1)e_6$ $e_2e_4 = (\alpha + 2\beta^2 - 3\beta + 1)e_6$ $\quad e_3e_1 = (2-\beta)e_5$ $e_3e_2 = ((\alpha-\beta)(3-2\beta)+1)e_6$ $\quad e_4e_1 = (\alpha - \beta + 1)e_5$ $e_4e_2 = (2\alpha - 2\alpha\beta + 2\beta - 1)e_6$ $\quad e_5e_1 = (2\alpha - 2\beta + 1)e_6$
$A^6_{11}(\beta)$	$\alpha = \frac{1}{2}\left(\beta - \sqrt{(-2+6\beta-3\beta^2)}\right), \beta \neq \frac{1}{2}$ $e_1e_1 = e_2$ $\quad e_1e_2 = e_4$ $\quad e_1e_3 = \alpha e_5$ $e_1e_4 = \beta e_5 + e_6$ $\quad e_1e_5 = (2\beta - 1)e_6$ $\quad e_2e_1 = e_3$ $e_2e_2 = e_5$ $\quad e_2e_3 = (2\alpha\beta - 2 + 1)e_6$ $e_2e_4 = (\alpha + 2\beta^2 - 3\beta + 1)e_6$ $\quad e_3e_1 = (2-\beta)e_5 - e_6$ $e_3e_2 = ((\alpha-\beta)(3-2\beta)+1)e_6$ $\quad e_4e_1 = (\alpha - \beta + 1)e_5 - e_6$ $e_4e_2 = (2\alpha - 2\alpha\beta + 2\beta - 1)e_6$ $\quad e_5e_1 = (2\alpha - 2\beta + 1)e_6$

Table A.5 – continued from previous page

A_{12}^6	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5 + e_6$
	$e_1e_4 = e_5 + 2\alpha e_6$	$e_1e_5 = e_6$	$e_2e_1 = e_3$
	$e_2e_2 = e_5 + \alpha e_6$	$e_2e_3 = e_6$	$e_2e_4 = e_6$
	$e_3e_1 = e_5$	$e_3e_2 = e_6$	$e_4e_1 = e_5 + (1-\alpha)e_6$
	$e_4e_2 = e_6$	$e_5e_1 = e_6$	
A_{13}^6	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5 + e_6$
	$e_1e_4 = \frac{3}{2}e_5$	$e_1e_5 = 2e_6$	$e_2e_1 = e_3$
	$e_2e_2 = e_5$	$e_2e_3 = e_6$	$e_2e_4 = 2e_6$
	$e_3e_1 = \frac{1}{2}e_5$	$e_3e_2 = e_6$	$e_4e_1 = e_6$
	$e_4e_2 = e_6$		
A_{14}^6	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_4 = \frac{1}{2}e_5$
	$e_1e_5 = 2e_6$	$e_2e_1 = e_3$	$e_2e_2 = e_5$
	$e_2e_3 = -3e_6$	$e_2e_4 = -2e_6$	$e_3e_1 = \frac{3}{2}e_5$
	$e_3e_2 = -3e_6$	$e_4e_1 = \frac{1}{2}e_5$	$e_4e_2 = e_6$
	$e_5e_1 = -4e_6$		
A_{15}^6	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_6$
	$e_1e_4 = \frac{1}{2}e_5$	$e_1e_5 = 2e_6$	$e_2e_1 = e_3$
	$e_2e_2 = e_5$	$e_2e_3 = -3e_6$	$e_2e_4 = -2e_6$
	$e_3e_1 = \frac{3}{2}e_5$	$e_3e_2 = -3e_6$	$e_4e_1 = \frac{1}{2}e_5 + e_6$
	$e_4e_2 = e_6$	$e_5e_1 = -4e_6$	$e_5e_1 = e_6$

Table A.5 – continued from previous page

A_{16}^6	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$
	$e_1e_4 = e_5$	$e_1e_5 = e_6$	$e_2e_1 = e_3$
	$e_2e_2 = e_4$	$e_2e_3 = e_5$	$e_2e_4 = e_6$
	$e_3e_1 = e_4$	$e_3e_2 = e_5$	$e_3e_3 = e_6$
	$e_4e_1 = e_5$	$e_4e_2 = e_6$	$e_5e_1 = e_6$
A_{17}^6	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$
	$e_1e_4 = e_5$	$e_1e_5 = e_6$	$e_2e_1 = e_3 + e_6$
	$e_2e_2 = e_4$	$e_2e_3 = e_5$	$e_2e_4 = e_6$
	$e_3e_1 = e_4$	$e_3e_2 = e_5$	$e_3e_3 = e_6$
	$e_4e_1 = e_5$	$e_4e_2 = e_6$	$e_5e_1 = e_6$
A_{18}^6	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$
	$e_1e_4 = e_5$	$e_1e_5 = e_6$	$e_2e_1 = e_3 + e_5$
	$e_2e_2 = e_4 + 2e_6$	$e_2e_3 = e_5$	$e_2e_4 = e_6$
	$e_3e_1 = e_4 + 3e_6$	$e_3e_2 = e_5$	$e_3e_3 = e_6$
	$e_4e_1 = e_5$	$e_4e_2 = e_6$	$e_5e_1 = e_6$